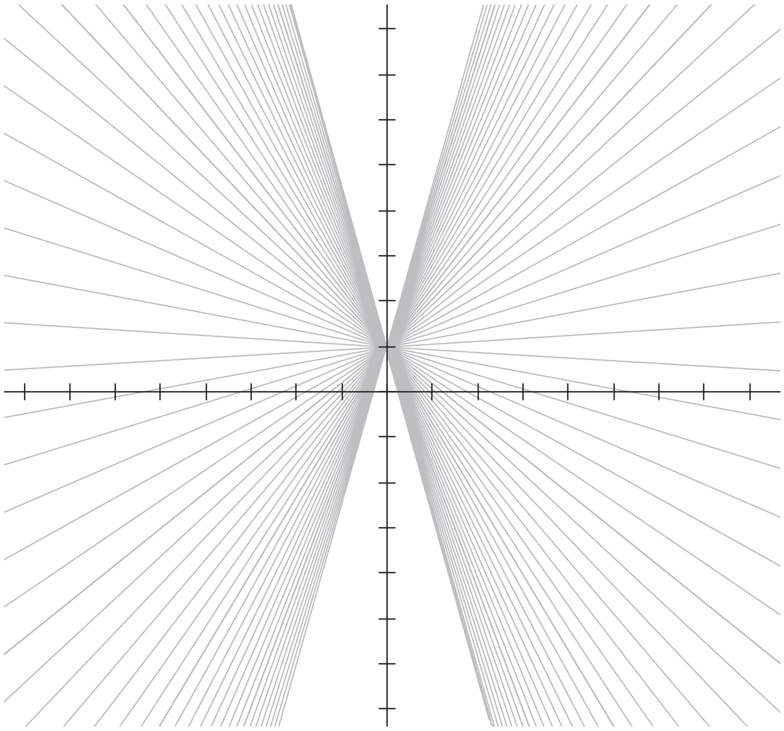
6

Variation and Linear Equations



Direct Variation

What happens to the area of a rectangle if you keep the length of the base constant while varying the height? (Try to answer this question before reading on.) What happens if you enlarge the entire rectangle? In this activity you will learn about direct variation and how it's represented algebraically and graphically.

SKETCH

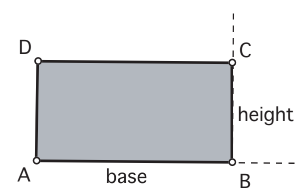
Start by constructing a rectangle and its interior.

1. In a new sketch, construct a point and label it *A*.
2. Translate point *A* by 1.0 cm at 0°. Construct a horizontal ray from *A* through the translated point.
3. Hide the translated point, construct a new point on the ray, and label it *B*.
4. Translate *B* by 1.0 cm at 90°. Construct a vertical ray from *B* through the translated point.
5. Hide the translated point, construct a new point on the ray, and label it *C*.
6. Mark the vector from point *B* to point *A*.
7. Translate point *C* by the marked vector. Label the translated point *D*.
8. Hide the rays and construct $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

To translate *A*, select it and choose **Transform | Translate.** To construct the ray, select the two points and choose **Construct | Ray.**

Select *B* and *A* in order and choose **Transform | Mark Vector.**

You should have a rectangle. Drag each of the four points to be sure it remains a rectangle and to see how dragging each point changes the rectangle.



9. Construct polygon interior *ABCD*. Measure its area.
10. Measure the lengths of base $\overline{AB}$ and height $\overline{BC}$ by selecting them and choosing **Measure | Length.**
11. Click the **Text** tool on $\overline{AB}$ and $\overline{BC}$ to show their labels. Change $\overline{AB}$'s label to *base* and $\overline{BC}$'s label to *height* by double-clicking the **Text** tool on each label.

Select the four points in order and choose **Construct | Quadrilateral Interior.**

INVESTIGATE

Next you'll investigate how the area changes as you change the size of the rectangle.

Q1 Drag the points around. How do the measurements change when you drag *B*? How do the measurements change when you drag *C*?

12. A graph will help to show what's happening. Select in order the *height* and *Area ABCD* measurements, and choose **Graph | Plot As (x, y).**

If the sketch is too cluttered, choose **Graph | Hide Grid.**

You have just plotted a point whose x- and y-coordinates are the height and area of the rectangle, respectively. (If you can't see it yet, you will soon.)

To trace an object, select it and choose **Display | Trace.** To clear traces from the screen, choose **Display | Erase Traces.**

13. Drag point C closer to $\overline{AB}$ and observe the effect on the plotted point. Trace the plotted point so you can see what it does as you drag point C.

Q2 What does the path of the plotted point tell you about how height and area are related in a rectangle when the base is kept constant?

Q3 Write the formula for the area of a rectangle in terms of *base* and *height.*

Q4 Now write the same formula as a function using $f(x)$ for *area* and x for *height.*

To enter *base* in your formula, click its measurement in the sketch. To enter *x,* click the *x* key on the Calculator's keypad.

Q5 Plot your function from Q4 by choosing **Graph | Plot New Function.** How does the function plot relate to the path of the plotted point as you vary the height of the rectangle?

Q6 Why does it make sense that the graph passes through the origin?

Q7 The function is plotted as a line. But the traces cover only part of the line. Why? If you wanted the function plot to accurately represent the situation, what part of it would you cut off? (In other words, how would you restrict the *domain*?)

Q8 We say that a rectangle's area *varies directly with* (or *is directly proportional to*) its height when its base is held constant. Describe in your own words what you think this means.

To answer Q10, compare a rectangle that produces a steep graph to one that produces a gradual graph.

Q9 Erase the traces. Drag point B to change the length of the base of the rectangle. What effect does this have on the graph?

Q10 How is the length of the base related to the slope of the graph?

EXPLORE MORE

Q11 Go to page 2 of **Direct Variation.gsp.** This page shows square $ABCD$. Does the area of the square vary directly with the length of one side? Make a prediction and then drag the points to check your prediction. Describe the path of the plotted point. Is this direct variation? Why or why not?

Objective: Students use a geometric model to study direct variation.

Student Audience: Algebra 1

Prerequisites: Students should be familiar with the concept of slope and be able to find the area of a rectangle.

Sketchpad Level: Intermediate. Students start with a new sketch and do a number of construction steps. They can skip many of these steps by starting with the prepared sketch.

Activity Time: 35–45 minutes. Starting with the prepared sketch will save 5 or 10 minutes.

Setting: Paired/Individual Activity (use a blank sketch, or **Direct Variation.gsp,** which shows the sketch after step 8) or Whole-Class Presentation (use **Direct Variation Present.gsp**).

In this activity students move from looking at properties of lines (in particular, slope) to generating linear relationships. Before starting, you might have a discussion about quantities that grow in different ways (proportionally, exponentially, and inversely) in relation to each other—and how each of these would look on a graph. The activity Inverse Variation focuses on quantities that are inversely proportional.

This activity can also be seen as a precursor to the activities that follow. Specifically, this activity focuses on lines of the form $y = mx$ (or $y = bx$), and a later activity (The Slope-Intercept Form of a Line) focuses on lines of the form $y = mx + b$. It would be valuable to make the connection here between direct relationships and linear relationships (of which direct relationships are a subset).

SKETCH AND INVESTIGATE

1. If Sketchpad is set to its default Preference settings, points won't be labeled when they are created. Students can click on points with the **Text** tool to label them. (Points will be labelled in alphabetical order.) To edit a label, double-click it with the **Text** tool.

Q1 Dragging B changes only the base and the area. Dragging C changes only the height and the area.

12. The sketch becomes cluttered at this point. Students may want to move the origin down near the bottom of the sketch window, hide the grid, and move the rectangle to a relatively clear area of the sketch.

Q2 It shows that as the height gets bigger, the area gets bigger; that as the height gets smaller, the area gets smaller; and that they grow or shrink proportionally to each other.

Q3 $A = base \cdot height$

Q4 $f(x) = base \cdot x$

Q5 It's the same. (To be more precise, it contains the path of the plotted point; the function exists in the first and fourth quadrants, but the plotted point is always in the first quadrant.)

Q6 The graph passes through the origin because the area of a rectangle with height 0 is 0—hence the point (0, 0). And algebraically, when $x = 0$ in $f(x) = base \cdot x$, $f(0) = 0$—hence the point (0, 0).

Q7 A rectangle can't have a negative height or area. The domain should be restricted to $x > 0$ (or possibly $x \geq 0$ if you consider 0 to be a possible height of a rectangle).

Q8 It means that as one quantity doubles, the other doubles; as one triples or halves, the other triples or halves. For example, the area of a rectangle with base 3 and height 4 is 12. If you double the height to 8 (and leave the base the same), the area also doubles to 24. The word "proportional" is used because the area and the height are in proportion ($12/4 = 24/8 = 3$). The base is the constant of proportionality.

Q9 This changes the slope of the line. Students may also notice that the plotted point moves vertically up and down (which makes sense because the x-value, which represents height, is not changing).

Q10 The length of the base is the slope of the graph. "Wide" rectangles (those with larger bases) will have steeper graphs. The reason is that every increase in height will add a lot to the area. "Skinny" rectangles (those with smaller bases) will have more gradual graphs. The reason is that similar increases in height will add much less to the area.

EXPLORE MORE

Q11 The point traces out a portion of a parabola. The trace is no longer linear, so this is *not* direct variation. The reason this happens is that we are now varying both the height and the base simultaneously, whereas before we were varying only the height, leaving the base constant. Variation in one dimension results in a linear graph, whereas variation in two dimensions results in a quadratic graph.

WHOLE-CLASS PRESENTATION

Use the sketch **Direct Variation Present.gsp** to explore with the class the relationships between measurements in a rectangle. The goal is to use this visual tool (and hopefully students' intuition) to think about why certain quantities are proportional and to connect this proportionality to direct variation via an equation and a graph.

Use page 1 of the sketch to answer Q1–Q10 together as a class. Use page 2 for Explore More (optional), which features a square instead of a rectangle. In this case, the area also increases when the side length increases, but the relationship is quadratic rather than linear.

Inverse Variation

Two variables, x and y, have an *inverse relationship* if y depends on the inverse of x:

$$y = \frac{k}{x}, \text{ where } k \text{ is a constant}$$

You can also express this in the form $xy = k$: The product of x and y is constant.

PUNCTUALITY

Adrianne has been late for school twice this week. She has resolved to become more punctual (and avoid detention). Her home is 3 miles from school, and the first class starts at 8:00. She figures that if she leaves the house at 6:30, she can walk to school at a leisurely pace and still arrive well before the bell. Before leaving, Adrianne does a quick calculation to see how fast she needs to walk. For this, she uses a formula that relates speed, time, and distance:

$$\textit{speed} \bullet \textit{time} = \textit{distance}$$

Q1 Is this an inverse variation? Which numbers are the variables, and which is the constant? Is it reasonable to call that number a constant?

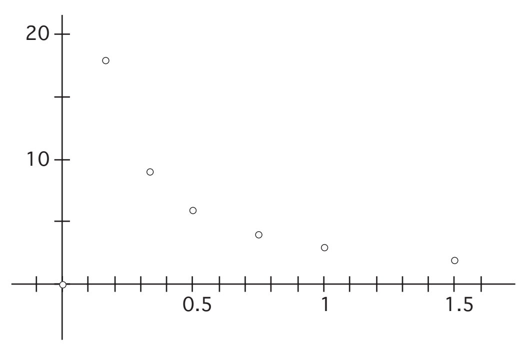
Let the x units be hours, and let the y units be mi/h. Do not use any other units.

Q2 Let x be the time (in hours) remaining before class, and let y be the speed that Adrianne needs to travel in order to arrive at 8:00. It is now 6:30. What is x and what is y? How fast must Adrianne walk?

Q3 While Adrianne was looking up the formula, 30 minutes passed. It is now 7:00. Compute x and y again. How fast must she walk if she leaves now?

It will have to be a fast walk. She decides to graph it, just to be thorough.

1. In a new sketch, choose **Graph | Grid Form | Rectangular Grid.** The rectangular grid allows you to adjust the x and y scales independently.
2. Choose **Graph | Plot Points.** Plot the (x, y) pairs that you computed in Q2 and Q3.



In the Plot Points dialog box, you can enter simple expressions. For example, enter 1/3 for one third of an hour (20 minutes).

Q4 The graph took longer than she expected. Now it's 7:15. How much time does she have left? How fast must she travel? Plot the point.

Q5 Now Adrianne has decided that it would be a better idea to ride her bicycle. There is still plenty of time if she leaves now (OK, as soon as this TV show is over). How fast will she have to ride if she leaves at 7:30? Plot the point.

Q6 Wouldn't you know it? Now it's raining. She is absolutely not going to school with wet hair. The bus will come by at about 7:40. How fast will it have to travel to get Adrianne to school on time? Plot the point.

Q7 Adrianne must have dozed. She's missed the bus. Now she will have to ride to school with her brother, Pete. He always drives too fast. He doesn't even leave the house until 7:50. How fast will he have to drive? Plot the point.

Q8 Look at the values you have found so far. In your own words, explain why the relationship between these two quantities is called *inverse variation.*

It's 7:50 now. Where's Pete? Adrianne's mother tells her that Pete took the bus. His car broke down yesterday. This is just too much! Now she'll have three tardy notices in one week, and it wasn't even her fault.

GRAPH THE CURVE

Adrianne now realizes that she was wasting too much time on calculations. Rather than compute each speed separately, she should have written *y* as a function of *x.* She could then plot the curve and find the speed for a whole range of departure times.

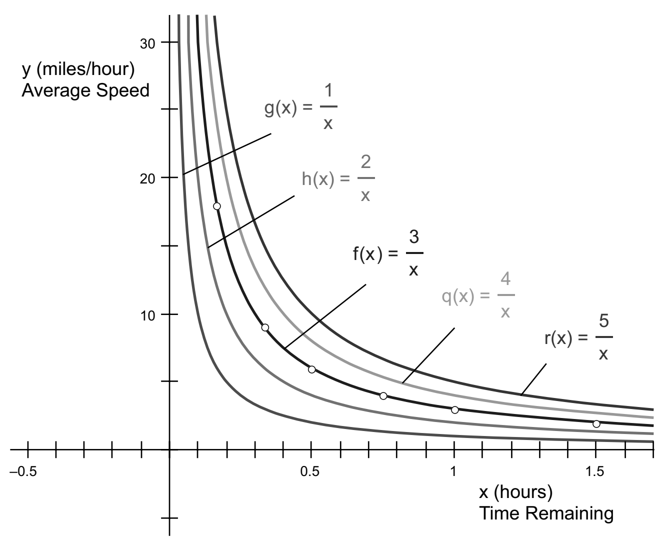
Q9 Write an expression for *y* in terms of *x.*

3. Choose **Graph | Plot New Function.** For the function definition, enter the formula for *y* from your answer to Q9.

Q10 What is the relationship between this curve and the points you plotted earlier?

To change the domain of a graph, select the graph and choose **Edit | Properties | Plot.** Negative time would make no sense here, so let zero be the minimum value of *x.*

4. This graph applies only to students who live 3 miles from school. Draw similar graphs for students living 1, 2, 4, and 5 miles from school. Use different colors. Label the axes and the curves.



Q11 Do any of these curves intersect? Explain why or why not.

Inverse Variation

Objective: Students plot (x, y) points representing an inverse relationship. Then they plot a family of curves.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None

Sketchpad Level: Intermediate. Students plot points and functions on a rectangular grid.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Inverse Variation Present.gsp**)

PUNCTUALITY

Q1 Yes, this is an inverse variation. Speed and time are variables, and the distance is a constant. In this context, distance is the distance from Adrianne's house to her school. That distance cannot change, so it is reasonable to call it a constant.

Q2 At 6:30, the time remaining (x) is 1.5 hours. The distance is 3 miles. Solving for y, Adrianne's average speed must be 2 mi/h.

$x = 1.5, y = 2$

Q3 She must walk 3 mi/h.

$x = 1, y = 3$

2. Check to see that everyone is plotting the first two points correctly, (1.5, 2) and (1, 3). After that, students should have no trouble plotting points on their own.

Q4 She has 0.75 hour remaining. She must travel 4 mi/h.

(0.75, 4)

Q5 She will have 0.5 hour remaining. She must ride her bicycle at 6 mi/h.

(0.5, 6)

Q6 She will have 1/3 hour remaining. The bus must travel 9 mi/h.

(1/3, 9)

It is possible to enter the fraction 1/3 in the Plot Points dialog box, or students can enter a decimal approximation.

Q7 There will be 1/6 hour remaining. Pete must drive 18 mi/h.

(1/6, 18)

Eighteen mi/h may sound slow for a car. Remind students that this is the average speed. Pete has to allow for traffic lights and stop signs, and he will have to find a parking space.

Q8 Answers will vary. Some students will observe that as one of the variables increases, the other decreases. Other students may be more precise, pointing out that the required speed is proportional to the multiplicative inverse of the remaining time.

GRAPH THE CURVE

Q9 $y = \frac{3}{x}$

Students may have already written the equation in this form. It would make it easier to answer the previous questions.

Q10 Since the curve represents the general solution for any given x, all of the plotted points should lie on the curve.

Q11 It is not possible for any two of the curves to intersect. If they did, the intersection point (x, y) would satisfy both equations. That means that the time and speed would be the same in both cases. If the time and speed are the same, then the distance traveled must be the same, but each curve represents a different travel distance.

This activity investigates the properties of an indirect variation in this form:

$$xy = k, \text{ where } k \text{ is a constant}$$

Tell students to imagine that they live 3 miles from school and that they must arrive there by 8:00. If they know how much time they have when they leave the house, they can compute what their average speed must be in order to arrive on time.

1. Write this formula: *speed* • *time* = *distance*

Q1 Ask students which number is a constant. It must be the distance because the distance between home and school does not change.

2. Define variables and units, and write them on the board. Write the equation too.

   x = time remaining before 8:00 (hours)

   y = speed (mi/h)

   $xy = 3$

3. Draw the following table, with only the top row filled in:

	6:30	7:00	7:15	7:30	7:40	7:50
x	1.5	1	0.75	0.5	0.33	0.17
y	2	3	4	6	9	18

4. Model the first column for them. It is 6:30, so she has 1.5 hours to get to class. Substitute 1.5 for x in the equation, and solve for y.

5. Have students fill in the rest of the table.

Tell them to stay with the given units—no minutes or seconds.

Q2 Ask students to explain why this relationship is called an inverse variation. (As x becomes smaller, y grows larger, and vice versa.)

6. Open **Inverse Variation Present.gsp.** There is a set of coordinate axes at an appropriate scale. Ignore the buttons for now.

7. Choose **Graph | Plot Points.** Enter the (x, y) coordinates from the table.

Q3 What is the equation for y as a function of x? $\left(y = \frac{3}{x}\right)$

8. Press the *Show Graph* button to reveal the graph of the equation. Confirm that all of the plotted points are on the curve.

Q4 Challenge students to describe what the general shape of the graph would be if the distance were 1 mile. (Same, but closer to the axes.) What if it were 5 miles?

9. Press the *Show Other Curves* button to see the graph for 1, 2, 4, and 5 miles.

Q5 Is it possible for any of these graphs to intersect? Discuss.

The Slope-Intercept Form of a Line

The slope-intercept form of a line, $y = mx + b$, is one of the best-known formulas in algebra. In this activity you'll learn about this equation first by exploring one line, then by exploring whole *families* of lines.

SKETCH AND INVESTIGATE

Choose **Graph | Define Coordinate System.** To hide the points, select them and choose **Display | Hide Points.**

You'll start this activity with $m = 2$ and $b = 1$ as you explore the line $y = 2x + 1$.

1. In a new sketch, define a coordinate system and hide the points (0, 0) and (1, 0).

Q1 For $y = 2x + 1$, what is y when $x = 0$? Write your answer as an ordered pair.

Choose **Graph | Plot Points.** Enter the coordinates in the Plot Points dialog box, click Plot, then click Done.

2. Plot this point. Why does it make sense to call this point the *y-intercept*?

Q2 You found that the y-intercept of $y = 2x + 1$ is 1. What is the y-intercept of $y = 3x + 7$? Explain why the y-intercept of $y = mx + b$ is always b.

You've learned that *slope* can be written as *rise/run.* The slope of the line $y = 2x + 1$ is 2, which you can think of as 2/1 (*rise* = 2 and *run* = 1).

3. Translate your plotted point using this slope. Choose **Transform | Translate,** use a rectangular translation vector, and enter 1 for the run (horizontal) and 2 for the rise (vertical).

To measure the coordinates, choose **Measure | Coordinates.**

Q3 What are the coordinates of the new point? Substitute them into $y = 2x + 1$ to show they satisfy the equation.

Q4 Translate the new point by the same *rise* and *run* values to get a third point. Find the coordinates of this third point, and verify that it satisfies the equation $y = 2x + 1$.

4. Select any two of the three points you've plotted, and choose **Construct | Line.**

What you've done so far is one technique for plotting lines in the form $y = mx + b$:

- Plot the y-intercept $(0, b)$.
- Rewrite m as *rise/run* (if necessary).
- Find a second point by translating the y-intercept by *rise* and *run.*
- Connect the points to get the line. Plot a third point to check the line.

If m is a decimal such as 1.5, write it as a fraction such as 3/2. If it's a whole number such as 3, write it as a fraction such as 3/1.

Q5 Using the method just described, plot these lines on graph paper.

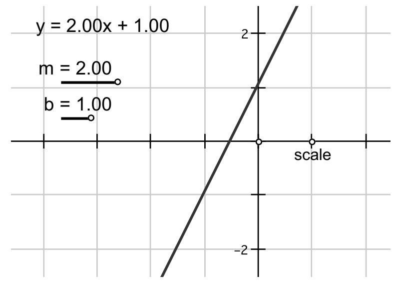
a. $y = 3x - 2$ b. $y = (2/3)x + 2$

c. $y = -2x + 1$ d. $y = 2.5x - 3$

EXPLORING FAMILIES OF LINES

Now that you've plotted a line, focus on how m and b affect the equation.

5. Open **Slope Intercept.gsp.**



The graph of $y = 2x + 1$ is already plotted. You can change m and b by adjusting their sliders.

To adjust a slider, drag the point at its tip.

Q6 Adjust slider m and observe the effect. Describe the differences between lines with $m > 0$, $m < 0$, and $m = 0$. What happens to the line as m becomes increasingly positive? Increasingly negative?

Q7 Now adjust slider b. Describe the effect this value has on the line.

6. Select the line and choose **Display | Trace Line.**

Q8 Adjust m and observe the trace pattern that forms. Describe the lines that appear when you change m. What do they have in common?

To erase traces left by the line, choose **Display | Erase Traces.**

Q9 Erase the traces and adjust b. How would you describe the lines that form when you change b? What do they have in common?

7. Turn off tracing by selecting the line and choosing **Display | Trace Line** again. Erase any remaining traces.

Q10 For each description below, write the equation in slope-intercept form. To check your equation, adjust m and b so that the line appears on the screen.

a. slope is 2.0; y-intercept is $(0, -3)$

b. slope is -1.5; y-intercept is $(0, 4)$

c. slope is 3.0; x-intercept is $(-2, 0)$

d. slope is -0.4; contains the point $(-6, 2)$

e. contains the points $(3, 5)$ and $(-1, 3)$

EXPLORE MORE

Q11 Attempt to construct a line through the points $(3, 0)$ and $(3, -3)$ by adjusting the sliders in the sketch. Explain why this is impossible. (Why can't you write its equation in slope-intercept form?)

Q12 Can you construct the same line with two different slider configurations? If so, provide two different equations for the same line. If not, explain why.

The Slope-Intercept Form of a Line

Activity Notes

Objective: Students explore the effects of intercept and slope on the position of a line. They practice writing equations in point-slope form and visualizing the graph when given the equation in point-slope form.

Student Audience: Algebra 1

Prerequisites: Students need to be familiar with the y-intercept and the *rise*/*run* definition of slope.

Sketchpad Level: Intermediate. Students plot and translate points and construct a line.

Activity Time: 25–35 minutes. The second part, Exploring Families of Lines, can be done on a different day, as Q5 is a good stopping point.

Setting: Paired/Individual Activity (use **Slope Intercept.gsp**) or Whole-Class Presentation (use **Slope Intercept Present.gsp**)

The value m is the slope of the line, and b is where the line crosses the y-axis. (This formula can also be written $y = a + bx$, using a and b instead of b and m. Some students may be more familiar with this form.)

SKETCH AND INVESTIGATE

1. Hiding the unit point (0, 1) reduces the chance that students will change the scale of the coordinate system. Sketchpad measures coordinates in graph units but does translation in distance (cm) units. When the coordinate system is defined, those units agree. If the points in Q3 and Q4 do not have integer values, the student has probably changed the scale by dragging the unit point or the tick numbers on the axes.

Q1 When $x = 0$, $y = 1$. The point is (0, 1). It makes sense to call this the y-intercept because it's the point where the line crosses the y-axis.

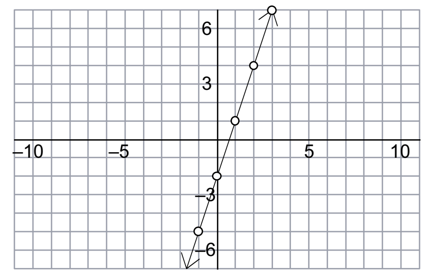
Q2 The y-intercept of $y = 3x + 7$ is 7. When you substitute 0 for x in $y = mx + b$, you get $y = m(0) + b$, or $y = b$.

Q3 The coordinates of the new point are (1, 3). This satisfies the equation because $y = 2(1) + 1 = 3$. (See the note for step 1 if students get non-integer coordinates when they measure them.)

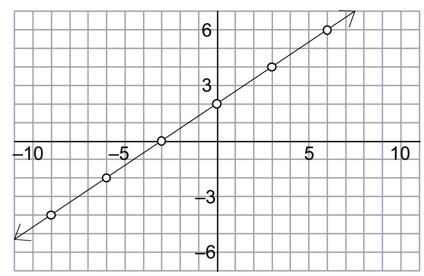
Q4 The third point is (2, 5). This point satisfies the equation because $y = 2(2) + 1 = 5$.

Q5 The lines are shown below with several integer points plotted.

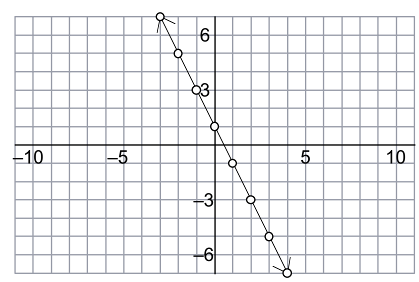
a.



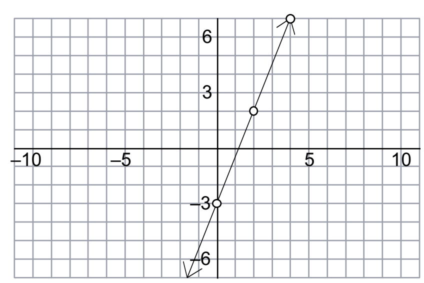
b.



c.



d.



Q6 Lines with a positive m go up to the right and down to the left, lines with a negative m go down to the right and up to the left, and lines with $m = 0$ are horizontal. As m becomes increasingly positive or negative, the line becomes steeper.

Q7 As b becomes increasingly positive, the line is shifted (translated) up. As b becomes increasingly negative, the line is shifted (translated) down. When $b = 0$, the line goes through the origin.

Q8 The slopes vary, but the traces always pass through the same y-intercept. The result looks like an "infinite asterisk."

Q9 This family can be pictured as the infinite set of lines in a plane that are parallel to a given line. They all have the same slope.

Q10 a. $y = 2x - 3$ b. $y = -1.5x + 4$

c. $y = 3x + 6$ d. $y = -0.4x - 0.4$

e. $y = 0.5x + 3.5$

EXPLORE MORE

Q11 This line is parallel to the y-axis, so it has no y-intercept and the slope is undefined. The line can be expressed with the equation $x = 3$, but that's not in slope-intercept form.

Q12 No, it's not possible. The reason is that every line has a unique y-intercept, so there's only one value for b for a particular line. Similarly, each line has a unique slope, so there's only one value for m.

WHOLE-CLASS PRESENTATION

Use the sketch **Slope Intercept Present.gsp** to help students visualize the graph of a line from an equation written in slope-intercept form. You will need to discuss how the y-intercept is found by substituting 0 for x, which always yields $y = b$ for an equation in the form $y = mx + b$. Then the slope can be applied to find one or two more points and graph the line.

Use page 2 to further explore the effects of m and b. This sketch is set up with sliders for m and b. You can use this sketch to explore Q6–Q12 with the whole class.

The Point-Slope Form of a Line

The slope-intercept form of a line is great if you know one special point: the y-intercept. But what if the point you know is an everyday, ordinary point such as $(3, -2)$ or $(-7, -7)$? In this case it's usually most convenient to use the *point-slope form* of a line, which you'll study in this activity.

SKETCH AND INVESTIGATE

1. Open the sketch **Point Slope.gsp.**

To adjust a slider, drag the point at its tip.

You'll see an equation in the point-slope form $y = m(x - h) + k$, with numbers filled in for m, h, and k. Adjust the sliders for m, h, and k, and watch the equation change. There's no line yet, but you can graph one.

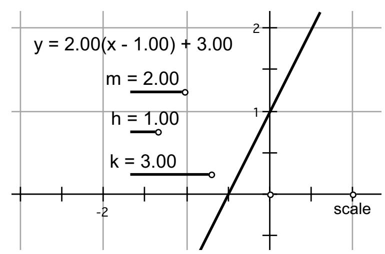
2. Choose **Graph | Plot New Function.**

The New Function dialog box appears.

To enter *m, h,* and *k,* click their measurements in the sketch. To enter *x,* click the *x* key on the keypad.

3. Enter $m(x - h) + k$ and click OK.

Sketchpad plots the function for the current values of the parameters m, h, and k.



4. Select the new line and choose **Display | Trace Function Plot.**

If at any point you wish to erase traces left by the line, choose **Display | Erase Traces.**

Q1 Adjust slider m. You'll see that the line rotates around a single point. Change the values of h and k, then adjust m again, focusing on where this point appears to be. What are the point's coordinates? How do they relate to h and k?

To deselect all objects, click the **Arrow** tool in empty space.

5. Deselect all objects. Now select, in order, measurement h and measurement k. Choose **Graph | Plot As (x, y)** to plot the point (h, k).

Q2 Adjust slider m again and observe what happens. Does this support your answer from Q1?

Q3 Describe the family of lines that forms when you change m.

Q4 Adjust sliders h and k, one at a time. How would you describe the families of lines that form when varying each of these values? How do they compare to each other?

Q5 Summarize the roles that h and k play in the equation $y = m(x - h) + k$.

Q6 Suppose you know that the slope of a line is 2 and that it contains the point $(1, 3)$. What is the equation in point-slope form for this line? Check your answer by adjusting the sliders in the sketch.

Q7 Write an equation in point-slope form for each of the lines described. When you finish, check each of your answers by adjusting sliders *m, h,* and *k* so that the line is drawn on the screen.

a. slope is 2; contains the point $(-2, 1)$

b. slope is -1; contains the point $(-2, 1)$

c. is parallel to the line $y = 3(x - 2) + 4$; goes through the origin

d. slope is $\frac{4}{5}$; x-intercept is $(2, 0)$

e. contains the points $(2, 3)$ and $(-1, 4)$

f. contains the points $(-3, 5)$ and $(4, 5)$

EXPLORE MORE

Q8 Try to construct a line through the points $(2, 3)$ and $(2, -2)$ by adjusting the sliders in the sketch. Explain why this is impossible and why this equation cannot be written in point-slope form.

Q9 Is it possible to construct the same line with different slider configurations? If not, explain why. If so, provide two different equations for the same line.

Objective: Students use Sketchpad's dynamic capabilities to examine the effect of each constant on a linear equation written in point-slope form. This activity helps to demystify the point-slope form.

Audience: Algebra 1

Prerequisites: Students should already have studied the slope-intercept form of a line.

Sketchpad Level: Easy/Intermediate. The hardest step is step 3, in which students plot the function (see step 3 below).

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (use **Point Slope.gsp**) or Whole-Class Presentation (use **Point Slope Present.gsp**)

SKETCH AND INVESTIGATE

3. Students may have to drag the New Function dialog box by its title bar in order to see measurements *m*, *h*, and *k* in the sketch. Students will need to enter the (implied) multiplication sign after *m*.

Q1 The line appears to spin around the point (h, k).

Q2 This does support the answer from Q1. You can now see the center point as the line spins.

Q3 This is the family of lines through the point (h, k) with any slope. This family can be pictured as an asterisk with the center point (h, k).

Q4 These are families of lines with the same slope. The families can be pictured as the infinite set of lines in a plane parallel to a given line. It's interesting that although the two families look the same, they are formed in different ways (as you can see by watching the point (h, k)). Adjusting *h* moves the lines right and left whereas adjusting *k* moves the lines up and down.

Q5 Parameter *h* is the *x*-coordinate of the special point that the line spins around when *m* is dragged. Making *h* larger moves the line to the right; making it smaller (or more negative) moves it to the left.

Parameter *k* is the *y*-coordinate of the special point that the line spins around when *m* is dragged. Making *k* larger moves the line up; making it smaller (more negative) moves it down.

Q6 $y = 2(x - 1) + 3$

Q7
a. $y = 2(x - (-2)) + 1$ or $y = 2(x + 2) + 1$
b. $y = -1(x - (-2)) + 1$ or $y = -1(x + 2) + 1$
c. $y = 3(x - 0) + 0$ or $y = 3x$
d. $y = 0.8(x - 2) + 0$ or $y = 0.8(x - 2)$
e. $y = (-1/3)(x - 2) + 3$ or $y = (-1/3)(x + 1) + 4$
f. $y = 0(x - 4) + 5$ or $y = 5$

EXPLORE MORE

Q8 Slope is defined as *rise/run*. For this line, since the *x*-coordinates of both points are the same (2), *run* = 0. Thus the slope is undefined, since you can't divide by 0. The line can be expressed with the equation $x = 2$, but that's not in point-slope form.

Q9 It is possible to express any line (except vertical lines) with an infinite number of equations in point-slope form (or an infinite number of slider configurations in the sketch). The reason is that *h* and *k* aren't unique—any point on the line will do. For example, the line $y = 2(x - 1) + 3$ also goes through the points (2, 5) and (3, 7), so the equations $y = 2(x - 2) + 5$ and $y = 2(x - 3) + 7$ also express this same line. (Try it out!)

WHOLE-CLASS PRESENTATION

Students see the effects of changing each of the values *m*, *h*, and *k*. They see how changing these values in a linear equation in point-slope form changes the location of the line on the graph. Use the sketch **Point Slope Present.gsp.** Press the button *Show Point on Line,* and drag x_p to show how the calculated *y* value changes for different values of *x*. Then press the button *Show Function and Line,* and drag x_p again. Use the sliders to change *m, h,* and *k,* and let students make observations. Do Q1–Q7 together as a class.

The Standard Form of a Line

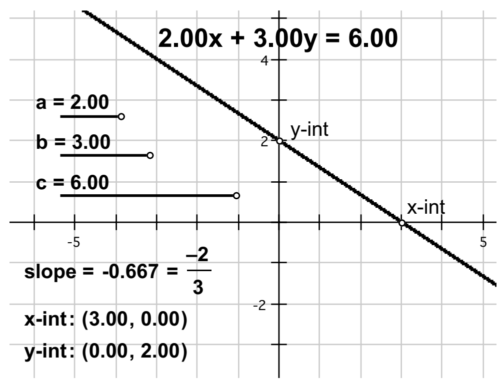
When he's in the form of Clark Kent, he's a mild-mannered reporter who helps expose government corruption; in the form of Superman, he's a superhero who can get your cat out of a tree or save Earth from obliteration. Same person, but different forms. Similarly, the equation of the same line can appear in different forms. And like Clark Kent/Superman, the different forms of the equation are useful in different ways.

You are familiar with linear equations in slope-intercept form $y = mx + b$ and point-slope form $y = m(x - h) + k$. In this activity you'll explore linear equations written in standard form $ax + by = c$. At first, this form may not seem to convey much useful information. But as you'll soon see, understanding this form can lead to a whole new way of looking at lines.

SKETCH AND INVESTIGATE

1. Open **Standard Form.gsp.**

Q1 Adjust the sliders *a*, *b*, and *c* (by dragging the point at the tip), and describe the effect each of them has on the line.



Select the both points and choose **Measure | Coordinates.**

2. Measure the coordinates of the *x*- and *y*-intercepts.

Q2 With slider *b* at 2.00 and slider *c* at 6.00, drag slider *a*. In a table like the one here, fill in the *x*-intercept for the given values of *a*.

a	2	3	12	−1	−3	0
***x*-intercept**						

In this activity, the precision of measurements has been set to two decimal places. Be aware that coordinate and slope measurements may have been rounded off.

Q3 What is the relationship between the *x*-intercept and the values of *a* and *c*?

3. Make $c = 12.00$. Adjust slider *b* and observe the effect this has on the line.

Q4 In a table like the one here, fill in the *y*-intercept for the given values of *b*.

b	2	3	12	–1	–3	0
***y*-intercept**						

Q5 What is the relationship between the *y*-intercept and the values of *b* and *c*?

4. Press the *Initial Position* button to return the sketch to its original state. Now adjust slider *c*, observing the line and the *x*- and *y*-intercepts.

5. Make $a = 2.00$ and $b = -4.00$. Once again, adjust slider c and observe the behavior of the line and the axis intercepts.

Q6 How do the values of a and b relate to the x- and y-intercepts of the line?

Q7 Based on your observations, write formulas for the x-intercept, y-intercept, and slope, using a, b, and c.

Q8 Find the equations of the following lines in standard form. First find the answers using paper and pencil; then check your answers by adjusting the sliders in the sketch.

a. parallel to $3x - 4y = 12$; x-intercept at $(1, 0)$

b. x-intercept at $(-5, 0)$; y-intercept at $(0, 2)$

c. x-intercept at $(4, 0)$; containing the point $(2, 1)$

EXPLORE MORE

Q9 Check your formulas from Q7 by algebraically manipulating the equation $ax + by = c$.

A *vector* is a quantity with both a distance and a direction, such as *3 units to the right.* A vector represented as [2, 1] goes 2 units right and 1 unit up. Vectors are often represented by arrows.

Q10 Another property of a and b is this: Vector $[a, b]$ is perpendicular to the line. To see this, press the *Show Vector* button. The blue segment represents vector $[a, b]$. The *foot* of the vector has been arbitrarily placed at the origin, and the *head* is represented by the arrowhead. Adjust all three sliders, and notice that the vector remains perpendicular to the line. Explain why $[a, b]$ is perpendicular to the line.

Q11 Find the equations of the following lines in standard form. First find the answers using paper and pencil; then check your answers by adjusting the sliders in the sketch.

a. perpendicular to the vector $[5, -3]$; passing through the origin

b. perpendicular to the vector $[-2, 4]$; passing through $(5, 3)$

Objective: Students use sliders to explore the effects of *a, b,* and *c* on the graph of a line expressed in the form $ax + by = c$. They look for relationships between these values and the slope and intercept values to understand the connection between different forms of the equation of a line. They also see some benefits of expressing a linear equation in standard form.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should know how to find slopes and intercepts. It helps if they are familiar with the slope-intercept and point-slope forms of a line.

Sketchpad Level: Easy

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (use **Standard Form.gsp**) or Whole-Class Presentation (use **Standard Form Present.gsp**)

After completing this and the other two "Form of a Line" activities, students should have a good understanding of the three forms and the information each provides. Point out that all three forms can describe a single line.

Start with a discussion on why there are so many forms. Begin with "Lines are very important. Mathematicians have developed different ways of describing them depending on the information they have and the information they need."

End this activity by asking students to compare and contrast three equivalent equations, such as $y = (2/3)x + 2$, $2x - 3y = -6$, and $y = (2/3)(x - 3) + 4$. Have students convert equations between forms and decide which form is most appropriate for a particular purpose.

SKETCH AND INVESTIGATE

Q1 Slider *a* affects only the *x*-intercept. (Students may also notice that positive values of *a* correspond to positive *x*-intercepts and negative values to negative *x*-intercepts. The relationship is inverse, so no matter how large *a* becomes, the *x*-intercept never reaches 0.)

Slider *b* affects only the *y*-intercept. (The relationship between *b* and the *y*-intercept is similar to the relationship between *a* and the *x*-intercept.)

Slider *c* changes only the position of the line. (It translates the line parallel to itself.) It has no effect on the slope of the line.

Q2

a	2	3	12	−1	−3	0
***x*-intercept**	3	2	0.5	−6	−2	undefined

Q3 The product of *a* and the *x*-intercept is *c*. This holds unless $a = 0$, in which case there is no *x*-intercept.

Q4

b	2	3	12	−1	−3	0
***y*-intercept**	6	4	1	−12	−4	undefined

Q5 The product of *b* and the *y*-intercept is *c*. This holds unless $b = 0$, in which case there is no *y*-intercept.

Q6 For all values of *c*, the intercepts are in the ratio $b{:}a$. Students may also note that when $c = ab$, the *x*-intercept is *b* and the *y*-intercept is *a*.

Q7 $x\text{-int} = c/a$; $y\text{-int} = c/b$; slope $= -(a/b)$

Q8 a. $3x - 4y = 3$ b. $-2x + 5y = 10$

c. $x + 2y = 4$

EXPLORE MORE

Q9 To check the formulas for the slope and *y*-intercept, convert the standard form to the slope-intercept form:

$$ax + by = c$$
$$by = -ax + c$$
$$y = -\frac{a}{b}x + \frac{c}{b}$$

Thus the slope is $-a/b$ and the *y*-intercept is c/b. To check the formula for the *x*-intercept, substitute 0 for *y*:

$$ax + by = c$$
$$ax + b(0) = c$$
$$ax = c$$
$$x = \frac{c}{a}$$

Q10 The vector remains perpendicular to the line because the vector's slope is b/a, and the slope of the line is $-a/b$. These quantities are negative reciprocals, so the vector and line are perpendicular.

Q11 a. $5x - 3y = 0$ b. $-2x + 4y = 2$

The Standard Form of a Line

Presenter Notes

In this presentation students will explore the graph of a line when the equation is in standard form ($ax + by = c$) and will see the effects of changing coeffecients.

1. Open **Standard Form Present.gsp.**
2. Introduce the activity and point out the equation displayed in standard form.

Q1 Ask how this equation looks different from linear equations expressed in point-slope and slope-intercept form. Here are some possible answers: x and y appear on the same side; y has a coefficient; and the slope does not appear explicitly in this equation.

Q2 Ask what the equation of this line would be in the other two forms. In slope-intercept form it would be $y = (-2/3)x + 2$. Multiple answers are possible in point-slope form. An answer using the point $(3, 0)$ is $y = (-2/3)(x - 3) + 0$, though infinitely many answers are possible using other points on the line.

3. Press the *Show Point on Line* button. Drag point P and observe the equation at the bottom. Explore how various combinations of x and y make the left side of the equation equal to the constant c on the right side.
4. Use the sliders to change a, b, and c, and explore the effects of these changes on the graph.

Q3 On page 2, complete the table by adjusting slider a to each specified value and double-clicking the table to record each value. Ask students to describe the relationship between the x-intercept and the values of a and c. (The x-intercept is equal to c/a.)

Q4 On page 3, complete the table by adjusting slider b, and ask students to describe the relationship between the y-intercept and the values of b and c. (The y-intercept is equal to c/b.)

Q5 On page 4, adjust slider c and ask students to describe its effect on the slope. (There is no effect.) Then adjust a and b to various positions, using whole numbers, and record the results by double-clicking the table. Ask students to describe the relationship between the two intercepts and the slope. (The slope is equal to $-a/b$.)

Q6 On page 5, press the buttons to show the various problems. For each problem, have students write down the answer, and then check their answers by manipulating the sliders.

Finish the presentation with a class discussion and summary of the relationship between the standard form coefficients and the characteristics of the line.

Lines of Fit

In science you often have to gather data from observations and use it to make a reasonable guess about something you cannot see. If you have a lot of data and it shows a consistent pattern, then you can have more confidence in your estimate.

PROCESSING DATA

Suppose you are an archaeologist, and your team has uncovered the remains of a shoe. The foot that fit the shoe would be 19 cm long, and by the type of footwear, you feel certain that it was a man's. In order to form a more complete picture of the shoe's owner, you would like to estimate his height. There are eight men working at the dig site. You decide to use them as your sample. You measure the foot length and height of each man. Here are your data:

Foot length (cm)	25	31	16	21	21	27	28	24
Height (cm)	189	195	149	174	158	169	180	172

Q1 There are also five women working at the site, but you do not measure them. Should you have included them in the sample? Explain.

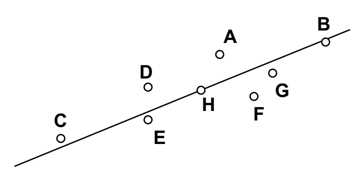
1. Open **Lines of Fit.gsp.** Notice that the horizontal and vertical scales of the coordinate axes are different.

To label all the points, choose **Edit | Select All** and then choose **Display | Label Points.**

2. Choose **Graph | Plot Points.** Enter 25 in the first box and 189 in the second, then click Plot. Repeat this for each of the ordered pairs. When you have plotted the last point, click Done. Label all the points.

BEST FIT-LINE

Q2 The simplest way (but not always the best way) to model the relationship between two variables is with a line. Is it possible to draw a line through the eight points on your screen?



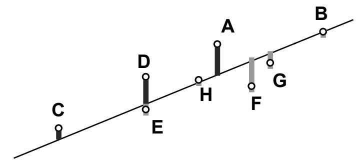
3. Choose the **Line** tool. Click in two places on the screen. When you do this, be careful not to click on any of the existing points or axes.
4. Choose the **Arrow** tool. By dragging the two points that define the line, move it into a position where you think it best fits the data points.

This line represents your best guess. If you were to plot the foot length and height of another man, you would exzpect it to fall somewhere near the line.

5. Select the line. Choose **Measure | Equation.**

Q3 What is the equation of your line?

6. Press and hold the **Custom** tools icon, and choose **|Residual|** from the menu that appears. Click on one of the data points and then click on the line. The tool plots a line segment and a measurement. Repeat this at each data point.



Q4 What is the meaning of the length and color of the line segments that are now attached to the data points? How are the measurements related to the line segments?

Perhaps you can adjust your line to fit the data better. The measurements are the magnitudes of the residuals. They tell you by how much each point misses the line vertically. In order to fit the data well, you need to minimize these values. The trouble is, if you move the line to make one smaller, you are probably making another one bigger.

7. Choose **Measure | Calculate.** Compute the sum of the magnitudes of the residuals. This is the total amount by which the line misses the data.

Q5 Adjust the line to minimize that last calculation. What is the new equation for the line? Compare this with your answer to Q3.

8. Construct a point on the line. Measure its coordinates.

Q6 Drag the new point along the line until the x-coordinate is about 19. Based on your analysis, what was the approximate height of the man who owned the shoe?

EXPLORE MORE

Using classmates as a data sample, do some measurements and find a best-fit line. There are many measurements that you could use—for example, hand span, arm span, and shoulder width.

Objective: Students construct a scatter plot of eight data points. They approximate a line of best fit, and use that to make an estimate.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should understand rectangular coordinates and linear equations. No background in statistics is required.

Sketchpad Level: Intermediate. Students use several Graph menu commands and one custom tool.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Lines of Fit.gsp**) or Whole-Class Presentation (use **Lines of Fit Present.gsp**)

PROCESSING DATA

Q1 Answers will vary. The important thing is that students support their answers with arguments. This is a good question for discussion.

2. To show point labels as they plot the points, students can choose **Edit | Preferences | Text** and set Sketchpad to show labels automatically for all new points.

BEST-FIT LINE

The regression line constructed in this activity is not a least squares regression. Students only have to understand that their objective is to minimize the difference between estimated height and actual measured height.

Q2 In order to fit a line through all of the points, they would have to be collinear, and clearly, they are not. (In fact, it would not be possible to fit any function graph to the points, because two of them have the same x value.) However, it *is* possible to draw a line that comes close to all the points.

Q3 Answers will vary. Example: $y = 2.9x + 103$

Q4 The length of a segment represents the amount by which a data point misses the line vertically. It is blue if the point is above the line and red if it is below. The measurements are the lengths of the line segments in coordinate units.

Q5 This should be about the same as the answer to Q3, but this second method is more reliable.

Q6 About 159 cm.

EXPLORE MORE

With students at different stages of physical development, this experiment can generate an interesting range of data. You could also use this data to take a closer look at the question in Q1. Create separate best-fit lines for girls and boys, and see if there is a significant difference. (Keep in mind that some students may be sensitive about having their body measurements taken. Modify the directions as needed.)

This problem involves an archaeologist who has uncovered a man's shoe, which would fit a foot of length 19 cm. She would like to get a rough estimate of the man's height. There are eight men and five women working at the dig site. She measures the foot length and the height of each man, but she does not measure the women.

1. Open **Lines of Fit Present.gsp.** Page 1 has a set of coordinate axes at an appropriate scale.
2. Select the table and choose **Graph | Plot Table Data.** This shows a scatter plot representing the foot lengths and heights of eight men.
3. Choose the **Line** tool to draw a line through the data plot. You must click in two places to define the line. Be careful not to click on any existing objects. You want both points to be independent points.

Normally, for such a rough estimate, you would simply draw the line so that it looks right. Adjust the line by dragging the two line points. When the class concurs on its position, find the equation and get an estimate.

4. Select the line. Choose **Measure | Equation.** Have students substitute 19 for x in the equation and solve for y.
5. How good was the line? Go to page 2. This has the same scatter plot. From each point, a vertical line segment represents the residual.

Q1 Ask what the line segments represent. For any foot length (x), the line estimates a man's height (y). Each line segment represents the difference between the actual height of a man and the estimate.

Q2 On the left side of the screen is a measurement labeled *total error.* This is the sum of the absolute values of the residuals. Tell students that this number represents the sum of the lengths of all of the colored line segments. Ask them how to use that number to improve the best-fit line. Help them understand that this number should be as small as possible.

6. Keeping an eye on the sum calculation, alternate between adjusting the two control points of the line to minimize the sum. Use the equation of the new line to make a new estimate of the man's height.

Encourage a discussion of the methods used. Was it right to exclude women from the sample? Should foot length and height have a linear relationship? How could the archaeologist improve the estimate?

It would not take long to conduct the similar measurements on the students.