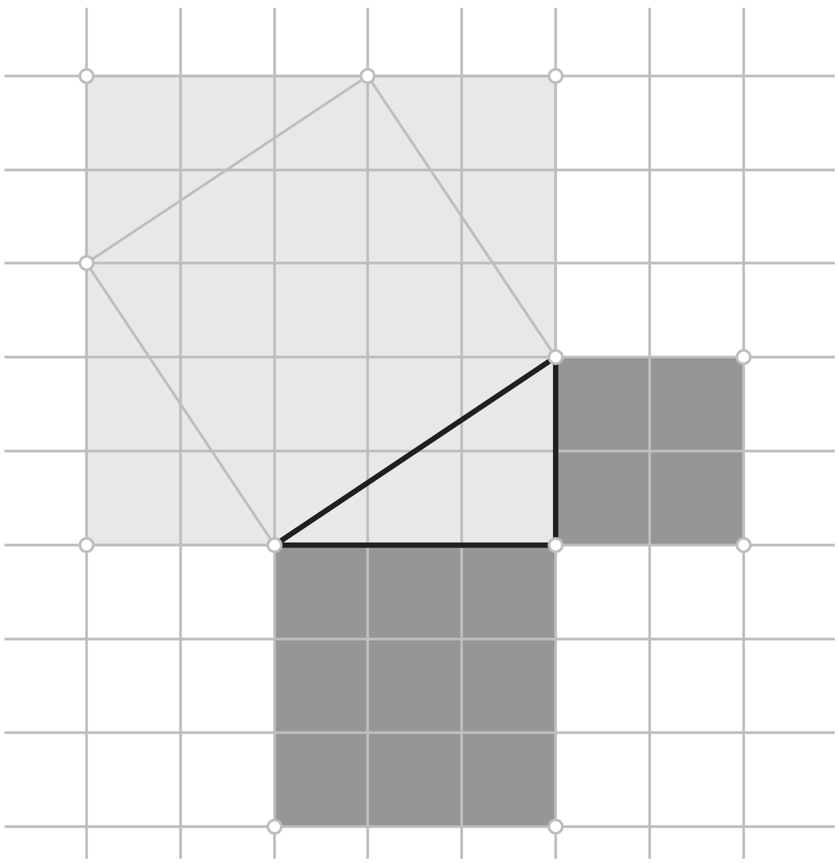
5

Coordinates, Slope, and Distance



Coordinates: The Fly on the Ceiling

Descartes is perhaps better known as a philosopher than as a mathematician. His most famous quote is *Cogito ergo sum*—"I think, therefore I am."

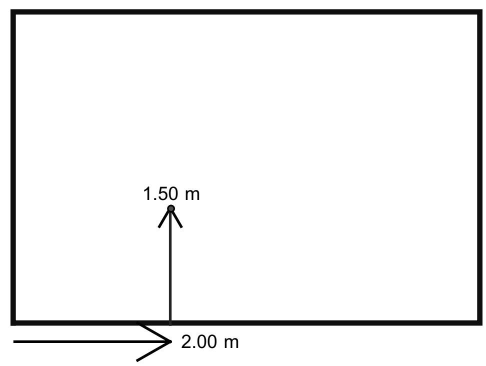
As the story goes, philosopher and mathematician René Descartes was gazing upward, deep in thought, when he saw a fly walking on the ceiling. It occurred to Descartes that he could describe the fly's position on the ceiling by two numbers: its distance from each of two walls. Thus was born the *coordinate plane,* also called the *Cartesian coordinate system* after Descartes.

In this activity you'll investigate the original idea of Descartes.

DESCARTES' FLY

Open **Coordinates.gsp.** You will see a model of the ceiling in Descartes' bedroom. Imagine that the red point is the fly in the story.

You can move the selected point in very small steps using the arrow keys on the keyboard.



x	y
2.00	1.50

1. Move the point and notice how the measurements change on the sketch and in the table to the right. These measurements are called *coordinates.*
2. Move the point to the upper right corner. Double-click the table to enter the measurements of the *x*- and *y*-coordinates.
3. Move the point to the other three corners and enter their coordinates into the table. Compare your four pairs of measurements with the measurements of other students.

Q1 How long and wide is the room? Explain how you can figure this out from the coordinates of the four corners.

4. Place the point as close as possible to the center of the room by eye.

Q2 What are the coordinates for the center point? Describe how you can figure this out using the coordinates of the four corners.

Q3 What are the coordinates for a point $\frac{1}{4}$ from the bottom and $\frac{2}{3}$ from the left?

PLOTTING POINTS

In this section you will plot points using their coordinates.

5. On page 2, plot these four points: (1, 1), (1, 3), (3, 1), and (3, 3). To plot them, choose **Graph | Plot Points.** For each point, enter its coordinates and click Plot. Click Done after you finish the last point.

Q4 Describe in detail the figure outlined by the points.

Q5 On page 3, select the table and choose **Graph | Plot Table Data.** Describe the figure outlined by the points.

You can find the coordinates of a point by selecting it and choosing **Measure | Coordinates.**

Q6 Page 4 shows two points that are opposite vertices of a square. What are the coordinates for the two missing vertices? Plot them.

Q7 The two points on page 5 are opposite vertices of a rhombus. Write down three pairs of possible coordinates for the two missing points. Plot the missing points.

Carpenters and other craftsmen use coordinates to mark panels for ceilings and walls so they can find water pipes and electrical wires hidden behind the panels. Before they cover a wall, they take notes on the coordinates for such hidden objects. Afterwards they can cut the necessary holes to install plumbing fixtures and electrical outlets and switches.

EXPLORE MORE

6. On page 6 there is a table with a set of coordinates. You will use them to decorate the ceiling in Descartes' room. Each ordered pair represents the position of a star in a well-known constellation.

Q8 Plot the table data. What is the name of the constellation?

Q9 Find a picture of your favorite constellation. Frame the picture in a rectangle, and measure the coordinates of each star with a metric ruler. Use page 7 of the document to plot the data one by one and make your own ceiling decoration.

You can start with a digital picture from the web. Use a search engine to find images.

Q10 You can even do this directly on the screen with Sketchpad. Copy any digital picture and paste it into page 7, or use the picture already in the sketch. Record in the table the positions of the edges of the objects in the picture. When you are done, hide the picture and plot the table data.

Coordinates: The Fly on the Ceiling

Objective: Students learn about rectangular coordinates by investigating Descartes' original inspiration for the system, a fly walking on a bedroom ceiling. They measure coordinates and plot points by coordinates.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None

Sketchpad Level: Intermediate. Students use a prepared sketch and a few menu commands, and collect data in and plot points from tables. All commands are explained when needed.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Coordinates.gsp**) or Whole-Class Presentation (use **Coordinates Present.gsp**)

Related Activity: The Origin: Center of the World

DESCARTES' FLY

1. Make sure students understand that the numbers in the table correspond to the metric measurements from the two walls to the point.
2. Here we introduce x and y as general names for the coordinates.
3. A class discussion based on the coordinate pairs of the four corners of the rectangle is a fine way to ensure that all students have understood the concept before moving on.

Q1 The room is 6 meters long and 4 meters wide. These are the ranges of the x- and y-coordinates.

Q2 The coordinates of the center of the rectangle are (3, 2).

Q3 The coordinates of the point are (4, 1).

PLOTTING POINTS

The amount of detail in students' geometric descriptions will depend on their geometry background.

Q4 The figure is a square with side length 2 m. Its perimeter is 8 m and its area is 4 m^2.

Q5 The figure is an isosceles trapezoid. The bases are 2 m and 4 m. The two other sides are each $\sqrt{5}$ m. The perimeter is $(6 + 2\sqrt{5})$ m and the area is 6 m^2.

Q6 The missing coordinates are (1, 3) and (3, 1).

Q7 There are many pairs of points that can be used to make the rhombus. The coordinates must be in this general form: $(3-a, 2)$ and $(3+a, 2)$, where $0 \le a \le 3$. Here are the possible solutions using integers only:

(0, 2) and (6, 2)

(1, 2) and (5, 2)

(2, 2) and (4, 2)

EXPLORE MORE

Q8 The constellation is the Big Dipper, one of the most distinctive constellations in the northern sky.

René Descartes, it is said, first got his idea for the rectangular coordinate system while he was pondering a fly that was walking on a ceiling. He knew that he could fix the fly's position in the plane with two numbers, the distances to two adjacent walls. Give students a clear picture of this situation, and perhaps some historical context.

1. Open **Coordinates Present.gsp.**
2. Drag the red point around the interior of the rectangle. Explain that the point represents the fly and the rectangle is the ceiling.

Q1 Ask what the two distances are measuring. They are the distances from the left edge and from the bottom edge.

3. Draw students' attention to the table at the right. These are the same distances written as coordinates. It is important that the students know which number is which.

Q2 Is it possible for one point to have more than one pair of coordinates? Is it possible for two different points to have the same pair of coordinates?

4. Drag the point to a corner of the ceiling. Double-click the table. This will save the coordinates of that point as a row in the table. Repeat at the other corners.

Q3 Ask the students to tell you the length and width of the ceiling (6 m by 4 m).

Q4 Ask what the coordinates would be at the very center of the room (3, 2).

5. Show how to plot points by coordinates. Choose **Graph | Plot Points.** Enter 3 and 2. Click Plot, then Done.
6. Go to page 2 and press *Show Constellation.* You will see an image of the constellation Leo. Drag the point to each star in turn, and double-click the table. After the last one, hide the constellation.

Explain that you have now recorded what you saw, and you can use the coordinates to reproduce the same image. In fact, anyone anywhere could do the same, even if they had never seen Leo.

7. Select the table. Choose **Graph | Plot Table Data.**

A good follow-up would be to have students draw patterns on graph paper, record the coordinates, and have someone else reproduce the image from the coordinates.

Q5 Ask students to name practical applications of coordinates that they have already used. Tell them that many digital computer images (GIFs, JPEGs) are just more sophisticated versions of the recording technique you used with the star constellation.

The Origin: Center of the World

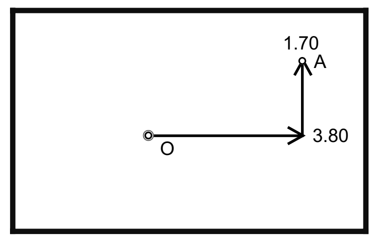
The word *origin* comes from Latin and means "place of birth."

In the previous activity we saw how Descartes measured the coordinates of a fly by noting its distance from each of two walls. It's possible to use instead a single fixed point (such as a light hanging in the middle of the room) as a reference point. We call this reference point the *origin,* and we use it as our chosen "center of the world."

CENTER OF THE WORLD

You can move the selected point in very small steps using the arrow keys on the keyboard.

Open **The Origin.gsp.** The enlarged point near the center of the sketch is the origin. You can describe the position of any other point by using the horizontal and vertical measurements from the origin.



x	y
3.80	1.70

1. Move point *A* and notice how the measurements change on the sketch and in the table to the right.

Q1 What happens to the *y*-coordinate when the position of the point is lower than the origin? And what happens to the *x*-coordinate when the point is to the left of the origin?

Add values to a table by double-clicking the table.

2. Move the point until it touches the right side of the rectangle. Add the *x*- and *y*-coordinate measurements to the table.
3. Do the same with the other three sides. Be careful to remember which row of the table corresponds to each side.

Q2 Use your measurements to find the length and width of the rectangle. Explain how you got your results.

Q3 Move the origin point to a new position, and use the same procedure to measure the dimensions of the rectangle. Did the size of the rectangle change? Explain your result.

BREAKING THE WALLS

4. Move the origin outside of the rectangle and use coordinates to measure the dimensions again.

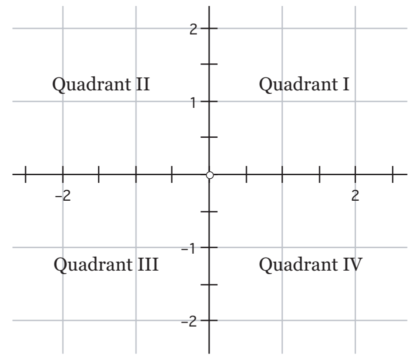
Q4 Did you find any difference? When using coordinates to measure distances, what difference does the location of the origin make?

5. Press the *Show Coordinate System* button and the *Hide Rectangle* button.

Always refer to quadrants with Roman numerals.

The axes are boundary lines and do not lie in any of the quadrants.

The x- and y-axes divide the plane into four regions called quadrants, numbered I, II, III, and IV, as shown here.



Q5 If a given point is in Quadrant I, what is the sign of its x-coordinate? What is the sign of its y-coordinate? For each of the quadrants, state a general rule about the signs of the coordinates of a point in that quadrant.

Q6 The origin for a surveying or mapping project is usually a point far away in a southwesterly direction. What do you suppose is the reason for choosing such a point for the origin?

INVESTIGATE MORE

You can change a parameter by double-clicking it, or you can select it and press the + and − keys.

6. Open the Target page of the same document. You can move the target (shown as a black cross) by changing the x and y parameters. Then move the arrow and leave a trace by pressing the *Go to Target* button. The arrow will go to the target, tracing its path as it does so. Practice changing the parameters and moving the arrow until you get a feel for the mechanics in the sketch.

To erase the traces, choose **Display | Erase Traces.**

Q7 Erase all traces before working on each of the following exercises. In each case, record the coordinates you use.

a. Draw a square.

b. Draw a triangle.

c. Draw a rhombus.

d. Make your own figure. It could be a star, a maze, or something different.

The Origin: Center of the World

Objective: Students investigate a coordinate system based on an origin, use negative coordinates, identify the four quadrants, and use coordinates to draw figures.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: No previous experience with coordinates is necessary. This activity will work best if it closely follows the activity Coordinates: The Fly on the Ceiling.

Sketchpad Level: Easy. Students manipulate a pre-made sketch, read measurements, and edit parameters.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **The Origin.gsp**) or Whole-Class Presentation (use **The Origin Present.gsp**)

Related Activity: Coordinates: The Fly on the Ceiling

CENTER OF THE WORLD

Q1 When the point is at a position lower than the origin, the *y*-coordinate is negative. When the point is to the left of the origin, the *x*-coordinate is negative.

Q2 The rectangle is 8 units long and 5 units wide. Find the length by subtracting the *x*-coordinate of a point on the left from the *x*-coordinate of a point on the right. Similarly, find the width by calculating the difference between the *y*-coordinates of points on the top and bottom.

Q3 The size of the rectangle is independent of the position of the origin. Thus the calculation should be the same as in Q2 even if the measured coordinates are different.

BREAKING THE WALLS

Q4 The dimensions of the rectangle should still be the same. Changing the position of the origin point makes no difference at all.

Q5 In Quadrant I, both coordinates are positive.

In Quadrant II, *x* is negative and *y* is positive.

In Quadrant III, both coordinates are negative.

In Quadrant IV, *x* is positive and *y* is negative.

Q6 If the origin is far to the southwest, all of the coordinates at the project site will be positive. This simplifies the calculations.

INVESTIGATE MORE

Q7 For this task, it is best to evaluate student progress by watching them as they work. Have them exchange coordinates to see if they can duplicate each other's work.

This is the final test. If students can draw these figures using coordinates, they probably have a solid understanding of the basics regarding the coordinate system.

IF TIME PERMITS

Students can finish this activity by working in pairs. First they should decide on a figure. It could be a hexagon, an arrow, or something else of their choice. They then discuss and decide which coordinates they should use to outline the figure. Finally, they can test their guess using the pre-made Target sketch.

This activity is a follow-up to Coordinates: The Fly on the Ceiling. Rather than measure the distances from two walls, you will use a single reference point, the origin.

1. Open **The Origin Present.gsp.** The biggest difference from the earlier activity is that the coordinates are now measured from an origin point in the middle of the rectangle, rather than using two walls for reference.

Q1 Drag point *A*, but keep it above and to the right of the origin. Ask the class what they expect to happen when you drag it to the left of the origin. The *x*-coordinate will become negative. Show this slowly. Show that the *y*-coordinate becomes negative when you drag it below the origin.

Q2 Tell the class that you want to measure the dimensions of the rectangle, and ask them for guidance. You can get the length by dragging point *A* to the right border, then the left, and comparing the *x*-coordinates. Check the *y*-coordinates at the top and bottom to get the width. To record coordinates of a point, drag point *A* into place and double-click the table.

2. Move origin point *O* to another location and repeat the measurements. Move it out of the rectangle and do the measurements once more. Check that students understand that changing the location of the origin will not change the measurements.

3. Press the *Hide Rectangle* and *Show Coordinate System* buttons. Review the names of the two axes and the four quadrants.

Q3 In what quadrant must point *A* be if both coordinates are negative? (III) Discuss the connections between the signs of the coordinates and the four quadrants.

4. Open the Target page of the document. The two coordinates control the position of the target (the cross). Press the *Go to Target* button. The arrow will travel to the target, leaving a trail as it moves. Demonstrate it by drawing a square with this sequence of points: (1, 1), (5, 1), (5, 5), (1, 5), (1, 1).

Q4 Get students to help you with coordinate plots of other figures. Here are some suggestions:

a. right triangle

b. isosceles triangle

c. rhombus

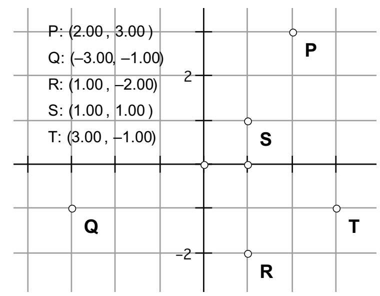
d. trapezoid

e. star

Points Lining Up in the Plane

If you've seen marching bands perform at football games, you've probably seen band members wandering in seemingly random directions suddenly spell a word or form a cool picture. Can you describe these patterns mathematically? In this activity you'll start to answer this question by exploring simple patterns of dots in the *x*-*y* plane.

SKETCH AND INVESTIGATE



1. In a new sketch, choose the **Point** tool from the Toolbox.

Holding down the Shift key keeps all five points selected.

2. While holding down the Shift key, construct five points.
3. With all points selected, choose **Display | Label Points.** Set the label of the first point to *P* and click OK.

To measure the coordinates, choose **Measure | Coordinates.**

4. Measure the coordinates of the five selected points.

A coordinate system appears, and the coordinates of the five points are displayed.

To hide objects, select them and choose **Display | Hide.**

5. Hide the point at the origin (0, 0), and the unit point (1, 0).
6. To make dragged points land only on locations with integer coordinates, choose **Graph | Snap Points.**

Q1 For each part a–h below, drag the five points to five different locations that satisfy the rule. Then copy your solutions onto the grids on the next page. Remember to fill in the coordinates of each point.

a. The *y*-coordinate equals the *x*-coordinate.

b. The *y*-coordinate is one greater than the *x*-coordinate.

c. The *y*-coordinate is twice the *x*-coordinate.

d. The *y*-coordinate is one greater than twice the *x*-coordinate.

e. The *y*-coordinate is the opposite of the *x*-coordinate.

f. The sum of the *x*- and *y*-coordinates is 5.

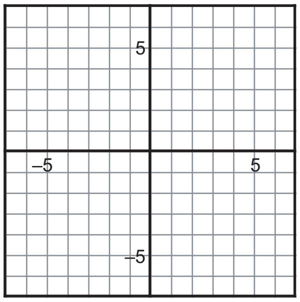
The absolute value of a number is its "positive value." The absolute value of both 5 and −5 is 5.

g. The *y*-coordinate is the absolute value of the *x*-coordinate.

h. The *y*-coordinate is the square of the *x*-coordinate.

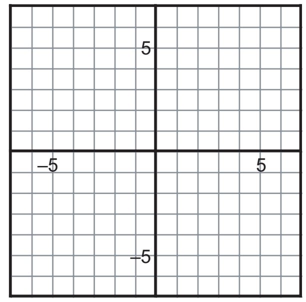
a. The y-coordinate equals the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



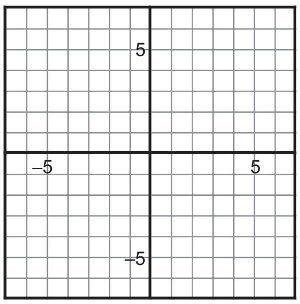
b. The y-coordinate is one greater than the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



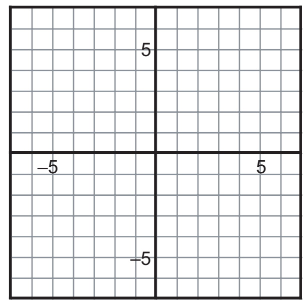
c. The y-coordinate is twice the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



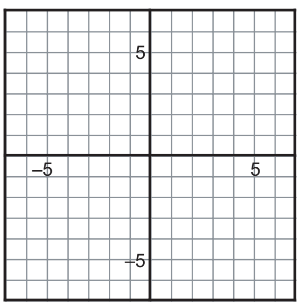
d. The y-coordinate is one greater than twice the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



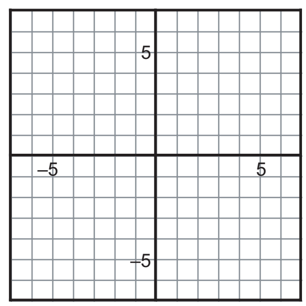
e. The y-coordinate is the opposite of the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



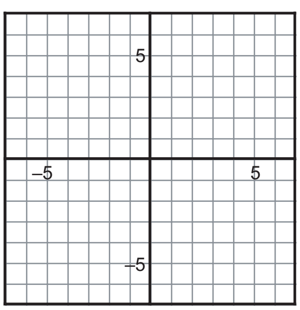
f. The sum of the x- and y-coordinates is 5.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



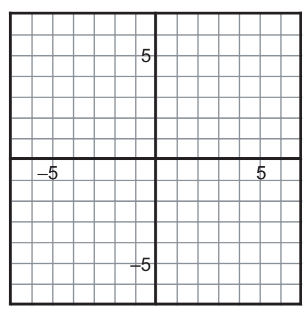
g. The y-coordinate is the absolute value of the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



h. The y-coordinate is the square of the x-coordinate.

P: ( )
Q: ( )
R: ( )
S: ( )
T: ( )



BACKWARD THINKING

In Q1, you moved points around to make them fit a certain rule. Here you'll reverse the process and be the detective. Your clues will be the given point positions, and your task will be to figure out the rule.

7. Open **Points Line Up.gsp.** You'll see a coordinate system with eight points (P–W), their coordinate measurements, and eight action buttons (a–h).

Q2 Press each action button in the sketch. Like the members of a marching band, the points will move about until they form a pattern. Study the coordinates of the points in each pattern, and then write a rule (like the ones in Q1) for each of the action buttons *a*–*h*.

EXPLORE MORE

Q3 Each rule in this activity can be written as an equation. For example, the rule for part b of Q1 ("The y-coordinate is one greater than the x-coordinate.") can be written as $y = x + 1$. Write an equation for each description in Q1 and Q2.

Q4 Add your own action buttons to those in **Points Line Up.gsp,** and ask your classmates to come up with descriptions or equations for your patterns. (Instructions on how to do this are on page 2 of the sketch.)

Q5 Go back to your first sketch. Turn off **Graph | Snap Points.** Drag each of your five points to a new location that satisfies the rule without using integer coordinates. Add your new points to grids a–h, and write their coordinates down on your paper.

Objective: Students find sets of points that satisfy algebraic rules and write algebraic rules to describe sets of points. This connection leads to a better understanding of coordinates, graphs, and equations (which they practice writing in the Explore More section).

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Familiarity with the Cartesian plane. The term *absolute value* is used and briefly defined, but it isn't a major focus of the activity.

Sketchpad Level: Easy. Students construct points and measure their coordinates.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (use **Points Line Up.gsp**) or Whole-Class Demonstration (use **Points Line Up Present.gsp**)

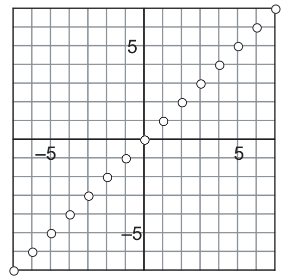
The purpose of this activity is to give students an informal and experiential introduction to the relationship between descriptions of coordinate patterns and graphs in the Cartesian plane. Too often, students don't really get the connection between an equation and its graph. It's important for them to understand that graphs depict the set of points whose coordinates satisfy an equation. This activity helps foster that understanding.

To deepen the experience, conduct a class or group discussion that encourages students to ponder this relationship. Ask, "Why do the points 'line up' in such regular ways? If you could plot not just five, but every point that satisfies the description, what would that look like?"

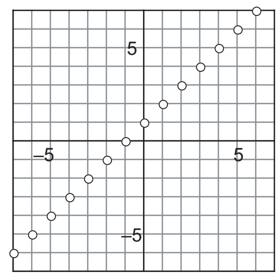
SKETCH AND INVESTIGATE

Q1 In each case, the answer shown depicts all possible answers with integer coordinates on the grid provided. The question asks for five answers, so any five of the points shown is a correct response (not to mention the infinite number of correct responses outside the grid!).

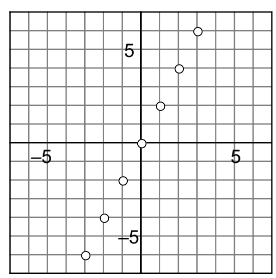
a.



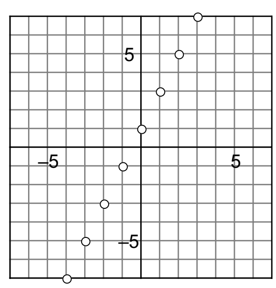
b.



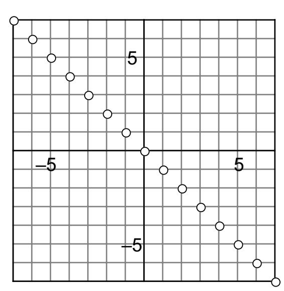
c.

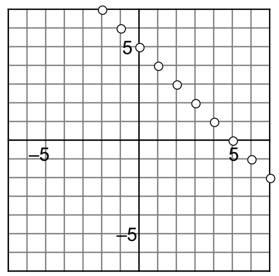


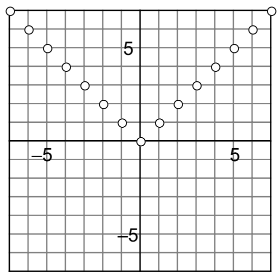
d.

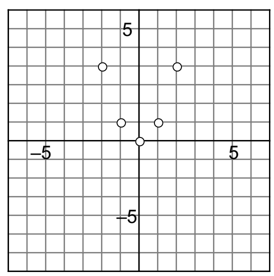


e.



f. 

g. 

h. 

Q2 a. The y-coordinate equals the x-coordinate.

b. The y-coordinate is one less than the x-coordinate.

c. The y-coordinate is twice the x-coordinate. (Or, the x-coordinate is one-half the y-coordinate.)

d. The y-coordinate is two less than twice the x-coordinate.

e. The y-coordinate is one-third the x-coordinate. (Or, the x-coordinate is three times the y-coordinate.)

f. The y-coordinate is always -1 (regardless of the value of the x-coordinate).

g. The y-coordinate is the opposite of the absolute value of the x-coordinate. (An acceptable alternate answer for students not familiar with the term *absolute value* might be "The y-coordinate is the 'negative value' of the x-coordinate, regardless of whether the x-coordinate is positive or negative.")

h. The product of the y-coordinate and the x-coordinate is 6.

EXPLORE MORE

Q3 Equations from Q1:

a. $y = x$ b. $y = x + 1$ c. $y = 2x$

d. $y = 2x + 1$ e. $y = -x$ f. $x + y = 5$

g. $y = |x|$ h. $y = x^2$

Equations from Q2:

a. $y = x$ b. $y = x - 1$ c. $y = 2x$

d. $y = 2x - 2$ e. $y = (1/3)x$ or $x = 3y$

f. $y = -1$ g. $y = -|x|$ h. $xy = 6$

Q4 Answers will vary.

Here's how to set up the Movement button (more detailed instructions are on page 2 of **Points Line Up.gsp**): Plot the eight destination points using the **Plot Points** command. Select all 16 points in the sketch in the following order: point P, point P's destination, point Q, point Q's destination, point R, point R's destination, . . . , point W, point W's destination. Now choose **Edit | Action Buttons | Movement.** Change the speed and label (on the Label panel), and then click OK. Now hide the eight destination points (using **Display | Hide**).

Q5 Answers will vary, but should line up with answers to Q1 and satisfy the rule.

WHOLE-CLASS PRESENTATION

Students connect verbal and graphical representations of points by using a verbal rule about coordinates to position points and by observing a pattern of points to formulate a verbal rule about their coordinates.

Use the sketch **Points Line Up Present.gsp** in conjunction with the Presenter Notes to present this activity to the whole class.

It's best to have a different student volunteer operate the computer for each rule.

1. Open **Points Line Up Present.gsp.**
2. Drag point *P* so that students can see how the coordinates change. Explain that students will take turns dragging the points to make the coordinates satisfy certain rules.

Encourage students to help each other in figuring out how to move the points and formulate rules.

3. Have the first student volunteer press button *a* to show the first rule, read the rule out loud, and then drag point *P* around until it satisfies the rule.
4. Have the student drag each of the remaining points around until all the points satisfy the rule.

Q1 Ask the class, "How would you describe the pattern these points make?"

Q2 Ask students to record on their paper both the rule and a diagram showing how the points are arranged.

5. Have a second student volunteer come to the computer, press button *b,* and drag the points for the second rule. Have students record on their paper each rule and a diagram of the resulting pattern. Continue for as many of the remaining rules as seems appropriate.
6. Go to page 2 of the sketch, and explain that on this page the points will arrange themselves and that the job of the class is to make up a rule that fits.
7. After students have written rules for all the arrangements on page 2, press the *a* button again to return the points to their initial arrangement.
8. Ask, "What rule did you write down for this arrangement?" After students have responded, ask "Does anyone know how to write this rule as an equation?" Make sure students understand why the answer is $y = x$.
9. Press each of the remaining buttons in turn, and have students give the equation for each pattern.
10. Tell students to go back to their answers for page 1 and write an equation for each of those arrangements.

Finish with a class discussion encouraging students to describe their insights. The discussion might consider questions such as these:

"How many points are there that would satisfy one of these rules?"

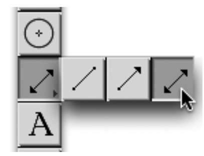
"If you could plot all the points that satisfy a rule, what would the result look like?"

"Why is it that the points line up so neatly?"

The Slope of a Line

The *steepness* of things—ski runs, wheelchair ramps, or lines in the *x*-*y* plane—can be described in lots of ways. For instance, skiers know that "black diamond" runs are steep and challenging, whereas "green circle" runs are less steep and easier. Mathematicians prefer to use numbers to describe steepness so that they can compare the steepness of objects and solve problems. In this activity you'll explore *slope,* a number that describes a line's steepness.

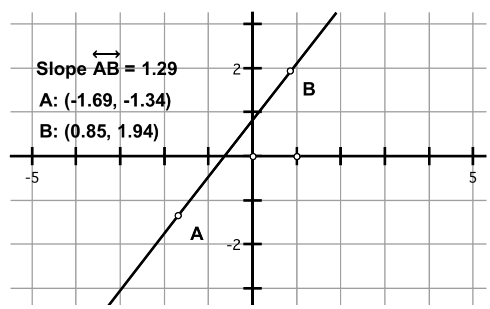
SKETCH AND INVESTIGATE



1. In a new sketch, press and hold the **Straightedge** tool in the Toolbox. Choose the **Line** tool from the menu that pops out.
2. Draw a line. Measure its slope by choosing **Measure | Slope.**
3. Measure the coordinates of the two points *A* and *B* that define the line.

Deselect all objects by clicking in blank space with the **Arrow** tool. Then select the two points and choose **Measure | Coordinates.**

4. Drag *A* and *B* to different locations and observe the changes in the slope measurement of the line.



Q1 Describe lines that have these slopes:

a. a slope of 1

b. a slope of −1

c. a slope of 0

d. an undefined slope

e. any positive slope

f. any negative slope

Q2 How do lines with a slope greater than 1 compare to lines with a slope of 1? How do lines with a slope between 0 and 1 compare to lines with a slope of 1?

Q3 Draw a new line. Drag it so that its slope is the opposite of the slope of your first line. How do lines having opposite slopes compare?

Be sure to drag the line itself and not either of its control points.

Q4 How does a line move when you drag it with the **Arrow** tool? What happens to the slope measurement when you drag a line this way?

5. Choose **Graph | Snap Points.** From now on, when you drag points, they will move only to locations with integer coordinates.
6. Display the slope to the nearest thousandth by selecting it and choosing **Edit | Properties.** Go to the Value tab and change the precision to thousandths.

Select the three measurements in order and choose **Graph | Tabulate.**

7. Create a table for the coordinates of *A*, the coordinates of *B*, and the slope.

A	B	Slope $\overleftrightarrow{AB}$
(0.00, 0.00)	(2.00, 4.00)	2.000

Q5 The table below shows different locations of *A* and values for the slope of $\overleftrightarrow{AB}$. Move point *A* to the indicated location; then find the coordinates of two locations for *B* that make the slope of $\overleftrightarrow{AB}$ equal the value in the last column. Each time you find a good location for *B*, double-click the table to "lock in" your entry.

You may need to make your sketch window larger to have enough space to find two answers to some of these problems.

If you have access to a printer, you can print out your table by printing the sketch.

A	*B*	*Slope* $\overleftrightarrow{AB}$
(0.00, 0.00)		2.000
(0.00, 0.00)		2.000
(2.00, 3.00)		−3.000
(2.00, 3.00)		−3.000
(−1.00, 4.00)		0.000
(−1.00, 4.00)		0.000
(2.00, −5.00)		1.250
(2.00, −5.00)		1.250
(3.00, 1.00)		Undefined
(3.00, 1.00)		Undefined
(−1.00, −3.00)		−2.667
(−1.00, −3.00)		−2.667
(−3.00, 2.00)		0.125
(−3.00, 2.00)		0.125
(4.00, 2.00)		3.500
(4.00, 2.00)		3.500

EXPLORE MORE

Q6 What is the relationship between the slope and the coordinates of *A* and *B*? Write down any patterns you notice, and use them to predict a third possible location for *B* for each slope and point *A* in the table. Use Sketchpad to check your prediction.

Q7 What happens if *A* and *B* trade places? Pick a row in the table and switch the locations of *A* and *B*. What happens to the slope? Why?

Q8 Pick any row in the table above. Turn off **Graph | Snap Points** and find some non-integer coordinates for *B*. Does this change your answer to Q6? How?

The Slope of a Line

Objective: Students measure the slope of a line and explore the relationship between the slope and the points that determine the line.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Familiarity with the Cartesian plane. Students should know what the origin is.

Sketchpad Level: Easy. Students do a simple construction and make several measurements.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Slope of a Line Present.gsp**)

The purpose of this activity is to get a general feel for slope before learning exactly how slope is calculated. The focus should be on qualitative relationships such as how all lines with a positive slope are different from all lines with a negative slope.

A good way to check if students have really internalized the concepts in this activity is to have them model lines using their forearms. Ask the class which arm would be more convenient for modeling lines with a positive slope (left) and which for those with a negative slope (right). Start by having students model slopes of 1 and –1, and remind them that these should be thought of as "points" of reference. Then call out slope values ("5 . . . −1 . . . 0.2 . . . −1/2 . . . 0 . . . −100"), and have students quickly approximate them with their arms.

An interesting discussion topic during or after this activity is whether there is a biggest or a smallest possible slope for a line.

SKETCH AND INVESTIGATE

3. The labels of the points should appear when students measure the slope. If not, students can click the **Text** tool on the points to show the labels. (You can choose **Edit | Preferences** and use the Text panel to control whether to show labels automatically as objects are measured. The normal setting is to show labels.)

Q1 a. A line with a slope of 1 will go up to the right and make angles of 45° with both axes.

b. A line with a slope of −1 will go up to the left and will also make angles of 45° with both axes.

c. A line with a slope of 0 is perfectly flat—horizontal, in other words.

d. A line with an undefined slope is perfectly vertical.

e. Lines with positive slopes always go up to the right and down to the left, regardless of how steep they are.

f. Lines with negative slopes always go up to the left and down to the right, regardless of how steep they are.

Q2 Lines with a slope greater than 1 are steeper than lines with a slope of 1. Lines with a slope between 0 and 1 are less steep than lines with a slope of 1. (It's helpful to think of lines with a slope of 1 as the "middle case" between steeper and less steep.)

Q3 They are just as steep, but in the opposite direction (one goes up to the right, the other up to the left). A fancy way of expressing this is that the two lines are reflections of each other across the vertical line through their point of intersection.

Q4 The line is translated—in other words, shifted in some direction—but not rotated, so its slope doesn't change, and neither does the slope measurement. One way of saying this is that the line is dragged "parallel to itself."

Q5 There are infinitely many solutions for each blank in the table. (Why?) Those listed here are the ones that fit in a normal-sized sketch window. Where fewer than three points fit, the three closest points—one in either direction—are listed.

A	*B*	*Slope* $\overleftrightarrow{AB}$
(0, 0)	(1, 2), (2, 4), (−1, −2), or (−2, −4)	2.000
(2, 3)	(3, 0), (4, −3), (1, 6), or (5, −6)	−3.000
(−1, 4)	(−2, 4), (0, 4), (1, 4), (2, 4), or (3, 4)	0.000
(2, −5)	(6, 0), (10, 5), or (−2, −10)	1.250
(3, 1)	(3, −2), (3, −1), (3, 0), or (3, 2)	Undefined
(−1, −3)	(−4, 5), (−7, 13), or (2, −11)	−2.667
(−3, 2)	(5, 3), (13, 4), or (−11, 1)	0.125
(4, 2)	(2, −5), (0, −12), or (6, 9)	3.500

EXPLORE MORE

Q6 Answers will vary. There are many possible relationships that students may notice. Some of these relationships are specific to particular slopes and positions of *A*. For instance, when *A* is at (0, 0) and the slope is 2, *B*'s *y*-coordinate is always twice its *x*-coordinate. At this early stage of exploring slope, it's more important to encourage careful observations than to develop the formal definition.

The effort to predict a new position of *B* from their observations helps students focus those observations.

Q7 The slope doesn't change when *A* and *B* exchange places, because they still determine the same line.

Q8 There are infinitely many non-integer coordinates for *B*. The points may follow similar patterns to the ones students observed in Q6, depending on what patterns they described.

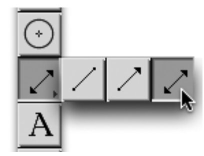
WHOLE-CLASS PRESENTATION

Introduce or revisit slope as a measure of steepness, and ask students to think about how they might assign a numeric value to describe it. Use the sketch **Slope of a Line Present.gsp,** and drag *A* and *B* to show how slope changes. Pose questions Q1–Q5 from the activity and elicit student participation. Using **Graph | Snap Points** in step 5 restricts the coordinates of *A* and *B* to integer values, making it easier for students to see how slope is mathematically related to these numbers.

The Slope Game

Imagine a game that combines the best elements of laser tag, Doom, and chess. This is not that game, but it is still a fun math game that's good for solidifying your sense of slope. The game works best with a partner (this is how it's described), but you can also play it alone if you hide the labels and cover the slope measurements before dragging the lines.

PLAYING THE SLOPE GAME



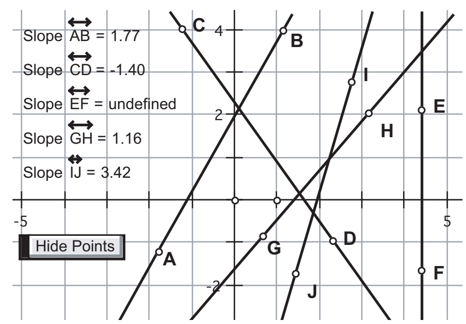
1. In a new sketch, press and hold the **Straightedge** tool in the Toolbox. Choose the **Line** tool from the menu that pops out.

Click and release only in blank space so that none of the lines are attached to each other.

2. Draw five different random lines in your sketch.
3. Measure the slopes of the five lines by selecting all of them at once and choosing **Measure | Slope.**

An easy way to select all the points is to choose the **Point** tool and then choose **Edit | Select All Points.**

4. Create a Hide/Show action button by selecting all the points and choosing **Edit | Action Buttons | Hide/Show.**



5. Press the button to hide the points.
6. Challenge your partner to match each measured slope with a line. Your partner must drag each measurement on top of the line it matches.
7. When your partner has finished guessing, press the button again to show the points. Your partner receives one point for each correctly matched slope.

You can scramble the lines automatically by selecting them and making an Animation button. Press the button to start scrambling the lines; press it again to stop.

8. Switch roles. While you look away, your partner will scramble the lines and hide the points. Now it's your turn to match the slopes with the lines.
9. After a round or two, you can add more lines to make the game more challenging.

The Slope Game

Objective: Students construct and play a game in which one player rearranges lines on the screen and the other player tries to match each line with its slope measurement.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should have had an introduction to slope, though it isn't necessary for them to know the rise/run definition yet.

Sketchpad Level: Easy. Students draw, drag, and measure the slopes of lines.

Activity Time: 5–15 minutes. This activity works well as a follow-up to the Slope of a Line activity.

Setting: Paired/Individual Activity (no sketch needed)

This simple, unassuming game has been a favorite in classrooms and workshops for years. Students really do enjoy trying to trick each other with lines that are very close to each other in slope, or the opposite of each other, and this represents a good learning opportunity.

PLAYING THE SLOPE GAME

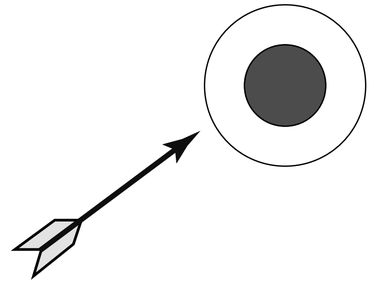
2. This step says to draw "five different random lines." The lines should not be attached to each other. In other words, students should click or release only in blank space when constructing the lines so that all the control points are independent points.

AFTER PLAYING

The More Slope Games activity contains four different ready-made slope games that can be played in a variety of ways. They can be played immediately after this game, but we recommend spreading them out over several days or weeks, letting students play each one for 5–15 minutes as time permits.

More Slope Games

Nothing improves your skills like practice. Open **More Slope Games.gsp.** The document contains four games designed to improve your understanding of slope. You can set the games up for one player or two.



GUESS THE SLOPE

Start the game with the *Play* button. You will see a random line. Edit the parameter *Guess Slope* by double-clicking it, making your best guess for the slope of the line, and pressing OK. Press the *Enter* button to see your score. You get one point if your line is within 10° of the correct slope and two points if you are within 5°. Be careful to get the sign of the slope right. Get it wrong and you will lose one point, no matter how close your guess is.

GUESS THE LINE

This is the same game in reverse. When you press *Play,* you will see a slope. Try to make the line match that slope by dragging the points. Scoring is the same as in Guess the Slope.

MATCH THE SLOPE

This game gives you five lines and five slopes. Drag the slope measurements onto their corresponding lines. You must place the center of the measurement on (or very close to) the line to get credit. This game is best if you play it as part of a two-person team competing against another two-person team.

SLOPE ARCHERY

The red point is an archer's shooting position. When you press *Play,* the target will move to a new random location on the border of the range. Change the *Guess Slope* parameter to match an imaginary line between the archer and the target. Press *Enter* to shoot. A hit scores one or two points. There is no penalty for misses, no matter how bad.

Objective: Students acquire an intuitive feel for slope by competing to associate lines with their corresponding slopes.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should be familiar with the slope concept. The activity does not require calculation of slope from coordinates.

Sketchpad Level: Easy

Activity Time: 5–10 minutes for ten rounds of one game with two players. There are four separate games in the activity.

Setting: Paired/Individual Activity (use **More Slope Games.gsp**)

Distributing copies of the student notes is not really necessary. There are no questions to answer. The rules are quite simple, and they also appear on the first page of the Sketchpad document itself.

With all these games, have students work in pairs if possible. A sense of competition will tend to keep them on task.

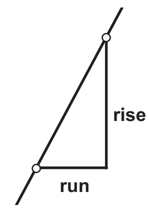
GUESS THE SLOPE

The only way to lose points is to guess the sign of the slope incorrectly. This is perhaps the most common mistake. Students tend to equate slope with steepness, disregarding the direction of the pitch. Encourage students to make a decision about the sign of the slope at the outset.

Students should also build an inventory of mental reference images. With practice, it does not take long to visualize lines with slopes of 1, 0, or −1.

GUESS THE LINE

A good tool to employ here is the rise/run formula for slope. Form an imaginary right triangle using the segment between the control points as the hypotenuse.



MATCH THE SLOPE

Students benefit most by playing this game as two-person teams, so that team members can work together to formulate strategies for comparing line slopes.

When matching a slope measurement to a line, drag the measurement so it is centered directly over the line. You score a point only if the center of the measurement is close to the line without being closer to another line.

SLOPE ARCHERY

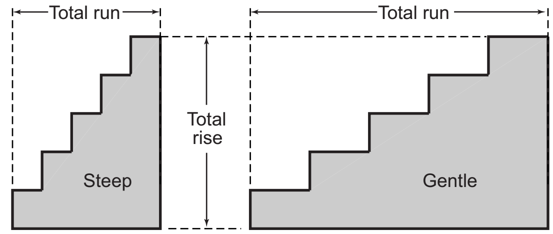
This is actually a variation of the Guess the Slope game, but with a higher degree of difficulty. Students must guess the slope of a line that is not even shown.

One predictable mistake will occur when the target is below and to the left of the archer. Students will tend to associate both of these directions with negative, but taken together, down and left produce a positive slope.

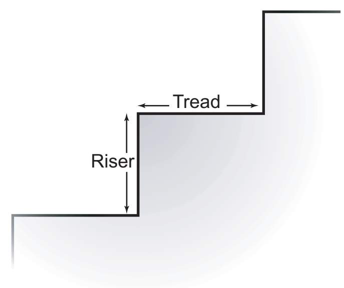
The Slope Archery game also involves an element of chance. The target's distance is variable because it is on the rectangular boundary of the archery range.

How Slope Is Measured

To build a staircase, contractors first need to determine the *total rise* (height) and the *total run* (length) of the staircase. If the total rise is large compared to the total run, the stairs will be steep (and dangerous!). If the total rise is small compared to the total run, the stairs will be easier to climb. So the relationship between the total rise and the total run determines the steepness of the staircase.

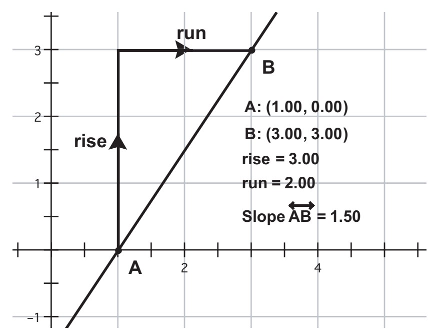


Each step also has a rise and run. The vertical part of a step is called the *riser,* and the horizontal part (where you step) is called the *tread.* A step with a large riser and small tread is steep. A step with a smaller riser and longer tread is safer and easier to climb. The steepness of each step depends on the overall steepness of the staircase. As you will see, this way of describing steepness is closely related to how slope is measured.



SKETCH AND INVESTIGATE

1. Open **Slope Measurement.gsp.** Press the *Show Coordinates* button.



Imagine building a staircase on this line, with one step going from point *A* to point *B*.

The rise is like the step riser and the run is like the tread. Press the *Show Staircase* button to see more stairs.

2. Press the *Step* button and observe the rise and the run. Drag *A* and *B*, and watch how these segments and values change.

To keep track of your results using the table in the sketch, press the *Show Table* button. Double-click the table each time you want to add the current measurements to the table.

Q1 For each row of the table, drag *A* and *B* to match the given values, and fill in the rest of the row. (The first row has been filled in for you.)

A: (x_A, y_A)	**B**: (x_B, y_B)	***rise***	***run***	***Slope*** $\overleftrightarrow{AB}$
(2, 1)	(4, 2)	1	2	0.5
(4, 0)	(5, 3)			
(−5, −1)	(−3, 4)			
(−5, 3)	(5, 4)			
(2, −3)	(,)	6	2	

Q2 In Q1, *B* was always above and to the right of *A*, and *rise* and *run* were always positive. What if *B* is below or to the left of *A*? Fill in the table to find out.

A: (x_A, y_A)	B: (x_B, y_B)	*rise*	*run*	*Slope* $\overleftrightarrow{AB}$
(2, 1)	(4, 0)			
(1, −1)	(0, 4)			
(−3, 6)	(−5, −1)			
(3, 5)	(,)	−3	−4	

Q3 What happens to *rise*, *run*, and *Slope* $\overleftrightarrow{AB}$ if you switch *A* and *B*? Try this for some of the table values above. Explain your results.

Q4 What happens to *Slope* $\overleftrightarrow{AB}$ when *B* is above and to the left of *A*? What happens when *B* is below and to the left of *A*? Why do you think this happens?

Q5 Fill in the following table with three other possible locations for point *B*.

If you have a printer and have kept your results in the Sketchpad table, you can use **File | Print** to print the sketch (including the table).

A: (x_A, y_A)	B: (x_B, y_B)	*rise*	*run*	*Slope* $\overleftrightarrow{AB}$
(1, 1)	(3, 2)	1	2	0.5
(1, 1)	(,)			0.5
(1, 1)	(,)			0.5
(1, 1)	(,)			0.5

Q6 Describe the locations for *B* that give a slope of 0.5. Explain why you think this happens. Move *A* to a different location. Does your explanation still work?

Q7 Looking back at your tables, you should notice a relationship between *rise*, *run*, and *Slope* $\overleftrightarrow{AB}$. Write a formula for *Slope* $\overleftrightarrow{AB}$ that uses *rise* and *run*.

Q8 Write a simple formula for *rise* that uses some or all of x_A, y_A, x_B, and y_B.

Q9 Write a simple formula for *run* that uses some or all of x_A, y_A, x_B, and y_B.

Q10 Rewrite your formula for *Slope* $\overleftrightarrow{AB}$ using x_A, y_A, x_B, and y_B.

EXPLORE MORE

Q11 So far, you've thought of *rise* as going up or down from point *A* and *run* as going right or left from there to point *B*. Would the slope be different if you went the other way? Press the *Show B to A* button. You'll see two new segments, *RISE* and *RUN*, going from *B* to *A*. Why is the slope the same whether you go from *A* to *B* along *rise* and *run* or from *B* to *A* along *RISE* and *RUN*?

Q12 In the activity The Slope of a Line, you learned that the slope of any horizontal line is 0 and the slope of any vertical line is undefined. Explain why this makes sense now that you know how slope is measured.

Objective: Students connect their intuitive sense of slope to specific calculations based on the coordinates of two points (the slope formula). They explore how the relative position of the two points is related to the value of the slope.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: It helps if students know some basics about slope, such as what the difference is between positive and negative slope, what a line with a slope of 1 looks like, and so on. One way to learn these basics is to do other slope activities such as, The Slope of a Line and The Slope Game.

Sketchpad Level: Easy. Students manipulate a prepared sketch.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Slope Measurement.gsp** in either setting)

Related Activities: The Slope of a Line, The Slope Game, and More Slope Games.

Prior to starting this activity, you may wish to have a discussion on how students could measure slope or steepness—of a hill or staircase, for instance. The objective isn't to get students to discover the rule, but rather to help them appreciate the problem.

Discuss $\overleftrightarrow{AB}$ versus $\overline{AB}$. Discuss why the slope of $\overline{AB}$ represents the slope of the entire line. Ask, "Can you have a steep step on a staircase that isn't steep? Or vice versa?" (No, because the steepness of each step is the same as the steepness of the staircase. You may wish to point out the similar triangles.)

Be sure to tell students whether to collect their data on paper or in the Sketchpad table. If you have printers, it may be convenient for students to collect their data in the sketch and print the sketch when they finish the activity.

SKETCH AND INVESTIGATE

Q1 Here is the completed table (with answers in bold):

(x_A, y_A)	(x_B, y_B)	*rise*	*run*	*Slope*
(2, 1)	(4, 2)	1	2	0.5
(4, 0)	(5, 3)	**3**	**1**	**3**
(−5, −1)	(−3, 4)	**5**	**2**	**2.5**
(−5, 3)	(5, 4)	**1**	**10**	**0.1**
(2, −3)	**(4, 3)**	6	2	**3**

Q2 Here is the completed table (with answers in bold):

(x_A, y_A)	(x_B, y_B)	*rise*	*run*	*Slope*
(2, 1)	(4, 0)	**−1**	**2**	**−0.5**
(1, −1)	(0, 4)	**5**	**−1**	**−5**
(−3, 6)	(−5, −1)	**−7**	**−2**	**3.5**
(3, 5)	**(−1, 2)**	−3	−4	**0.75**

Q3 Switching *A* and *B* makes no difference, since the line and the step are still the same.

Q4 When *B* is above and to the left of *A*, the slope is negative; when *B* is below and to the left of *A*, the slope is positive. If *rise* and *run* have different signs, the slope is negative. If they have the same signs, the slope is positive.

Q5 Here are all possible integer answers in the original sketch window (with answers in bold):

(x_A, y_A)	(x_B, y_B)	*rise*	*run*	*Slope*
(1, 1)	(3, 2)	1	2	0.5
(1, 1)	**(5, 3)**	**2**	**4**	0.5
(1, 1)	**(7, 4)**	**3**	**6**	0.5
(1, 1)	**(−1, 0)**	**−1**	**−2**	0.5
(1, 1)	**(−3, −1)**	**−2**	**−4**	0.5
(1, 1)	**(−5, −2)**	**−3**	**−6**	0.5
(1, 1)	**(−7, −3)**	**−4**	**−8**	0.5

Q6 They are all on the same line. They can be reached by starting from A and repeatedly moving up 1 unit and right 2 units or down 1 unit and left 2 units.

Q7 $slope = rise/run$

Q8 $rise = y_B - y_A$

Q9 $run = x_B - x_A$

Q10 $slope = (y_B - y_A)/(x_B - x_A)$

EXPLORE MORE

Q11 The slope is the same in either direction. If you go the opposite way, the rise and the run will be the opposite of what they were before, and the ratio will be the same.

Q12 For a horizontal line, $rise = 0$, and $slope = 0/run$, which is 0 for any value of *run.* For a vertical line, $run = 0$ for any value of *rise,* and $slope = rise/0$, which is undefined.

WHOLE-CLASS PRESENTATION

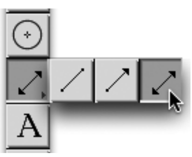
Use the sketch **Slope Measurement.gsp** to show how slope changes as you change the line. The *Step* button gives students a picture of rise and run as a "step." Ask students to visualize the subsequent steps on this staircase, and use the *Staircase* button to show them.

Slopes of Parallel and Perpendicular Lines

It's often important to know whether two lines are parallel or perpendicular. Can you figure this out from the slopes? In this activity you'll find out.

SKETCH AND INVESTIGATE

1. In a new sketch, choose **Edit | Preferences.**
2. On the Units panel, set Angle Units to **degrees,** Angle Precision to **hundredths,** and the precision for slopes and ratios to **thousandths.** On the Text panel, turn on the setting to show labels automatically for all new points. Click OK.



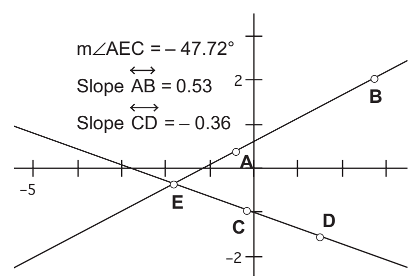
3. Press and hold the **Straightedge** tool until a menu pops out. Choose the **Line** tool from the menu.
4. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ so they intersect on-screen.
5. Construct their point of intersection E by clicking the intersection with the **Arrow** tool.

Select points A, E, and C in order. Then choose **Measure | Angle**.

6. Measure $\angle AEC$.
7. Measure the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ by selecting them and choosing **Measure | Slope.**

A coordinate system and the slope measurements appear.

8. Choose **Graph | Hide Grid.**
9. Drag various points and lines in the sketch and observe the measurements.



Q1 Make sure that neither line is horizontal or vertical, and make the slopes as close to equal as you can. What do you observe about the measure of the angle between the lines?

Q2 What can you say about lines with equal slopes? How can you verify your statement?

Choose **Measure | Calculate** to open the Calculator. Click on a measurement in the sketch to enter it into the calculation.

10. Calculate the product of the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Q3 Drag various objects and observe the product measurement. What does this value tell you about the two lines? Do any particular values seem to have special importance?

Q4 Make sure that neither line is horizontal or vertical. Now drag points to make $m\angle AEC$ as close to 90° as you can. What does the product of the slopes of perpendicular lines seem to be?

It's important to try more than one arrangement and see if your results agree.

Q5 Drag the points to make a different arrangement in which $m\angle AEC$ is as close to 90° as possible. What is the product of the slopes this time?

Q6 Why do you think the product in Q4 and Q5 is negative?

Q7 Why do you think Q4 says that neither line should be horizontal or vertical? Write down your conjecture first, and then drag the points to test it.

EXPLORE MORE

Select the line and a point to construct a perpendicular. Select the line and a point *not* on the line to construct a parallel.

Q8 So far, you have made observations based on approximate measurements. In a new sketch, draw a line and use **Construct | Perpendicular Line** and **Construct | Parallel Line** to construct lines parallel and perpendicular to it. Then measure the slopes and calculate the product. Is this more convincing than what you did before? Why?

A vector consists of a distance and a direction, such as 30 ft to the right. The vector from point *A* to point *B* is the distance and direction you would have to go to get from point *A* to point *B*.

Q9 In your original sketch, select points *A* and *B*, and choose **Transform | Mark Vector.** Select point *C* and choose **Transform | Translate;** click OK to translate by the marked vector. Now select the new point (C') and point *D*, and choose **Edit | Merge Points.** The two lines now have the exact same slope. What happens to the angle measurement? (Use **Edit | Undo Merge Points** and **Edit | Redo Merge Points** to watch it again.) Why do you think this happens?

Q10 Explain why saying that the product of the slopes of perpendicular lines is -1 is the same as saying that the slopes are *negative reciprocals* of each other. (Look up the word *reciprocal* if necessary.)

Slopes of Parallel and Perpendicular Lines — Activity Notes

Objective: Students construct two lines, measure the slopes of the lines and the angle between them, and draw conclusions about the slopes of parallel and perpendicular lines.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should understand the terms *parallel, perpendicular,* and *slope.*

Sketchpad Level: Intermediate. Students measure slopes and angles, set precision, and use the Calculator to find a product. (In Explore More, they also use the Construct and Transform menus.)

Activity Time: 15–25 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Slopes Parallel Perpendicular Present.gsp**)

You may wish to initiate this activity with a simple question, such as "How can you tell whether two lines are parallel? How about perpendicular?" Students may think of physical tools, and you can get them thinking in terms of the coordinate plane. Even if students do hit upon the idea of using slope, they're unlikely to know exactly how to use slope to test for perpendicularity.

The last Explore More item points out that slopes of perpendicular lines are negative reciprocals. Students may need to look up *reciprocal;* they may also know these as *opposite reciprocals* rather than *negative reciprocals.*

SKETCH AND INVESTIGATE

Q1 The angle measurement approaches 0° (and may even disappear when the slopes are precisely equal).

Q2 If two lines have equal slopes, then the lines are parallel. To verify this, you might look at many pairs of lines, or construct parallel lines using **Construct | Parallel Line** and make the necessary measurements.

Q3 Answers will vary. For example, if the product is positive, either both lines have positive slopes or both have negative slopes. Also, if the product is 0, at least one line is horizontal, with a slope of 0.

Q4 The product of the slopes of perpendicular lines is always −1 as long as neither line is vertical. Students are unlikely to be able to make the angle exactly 90°, so their results may be off by one or two hundredths.

Q5 For any arrangement not involving a vertical line, the product is still very nearly −1.

Q6 If two lines are perpendicular, one of the slopes must be positive and the other negative. The product of a positive and a negative number is always negative.

Q7 If one line is horizontal, the line perpendicular to it will be vertical. Although a horizontal line and a vertical line are perpendicular, the slope of a vertical line is undefined, and the product of an undefined quantity with any other number is also undefined.

EXPLORE MORE

Q8 With lines constructed to be exactly perpendicular, the slope product calculation should always be exactly –1, regardless of the precision used. (The only exception to this is when one of the lines is vertical and so has an undefined slope.) This should be more convincing than the earlier demonstration.

Q9 The angle measurement disappears when students merge the two points. The lines have the same slope because $\overleftrightarrow{CC'}$ runs in the same *direction* as $\overleftrightarrow{AB}$ (*not* because the distances CC' and AB are equal). The angle measurement disappears because intersection E no longer exists when the lines are parallel and therefore don't intersect.

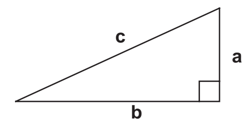
Q10 Reciprocals are "flipped" fractions, such as 2/3 and 3/2 or 5 and 1/5 (thinking of 5 as 5/1). The product of a number and its reciprocal is always 1. The product of a number and its negative reciprocal can be represented as $(a/b) \cdot (-b/a)$, which equals −1.

WHOLE-CLASS PRESENTATION

Students observe the slopes of two lines as they approach being parallel, then perpendicular. They also watch the product of the slopes as the lines approach perpendicularity and observe that the product approaches −1. Use the sketch **Slopes Parallel Perpendicular Present.gsp** and drag the points or press the buttons for a quick demo. Answer Q1–Q7 together as a class.

The Pythagorean Theorem

The Pythagorean theorem says that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse. For the right triangle shown here, you can write the Pythagorean theorem as an equation:



$$a^2 + b^2 = c^2$$

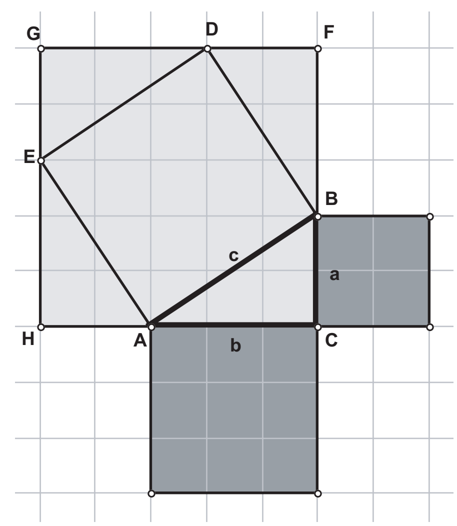
To prove the theorem, you must show that it is true for all right triangles. In this activity you will demonstrate the theorem for several specific cases.

COMPARING SQUARES

1. Open **Pythagorean Theorem.gsp.** You see right triangle *ABC* on a square grid.

Don't use **Measure | Distance** or **Measure | Area** in this activity, because you do not know the scale of the grid.

Q1 How long are *a* and *b*? What is the sum of the squares of the lengths of the legs ($a^2 + b^2$)? Don't measure—just count grid squares.



2. Press the *Show Leg Squares* button.

Q2 Explain why the areas of these squares are a^2 and b^2.

3. Press the *Show Hypotenuse Square* button.

This square, *ABDE,* has an area of c^2. Unfortunately, it is not aligned with the grid, so finding its area is more difficult.

4. Press the *Show Circumscribed Square* button.

Q3 This square, *CFGH,* fits around the hypotenuse square. What is its area?

Q4 There are four right triangles that also fit into the big square: $\triangle ABC$, $\triangle BDF$, $\triangle DEG$, and $\triangle EAH$. What is the area of each triangle? What is the sum of the triangle areas?

Q5 Use your answers to Q3 and Q4 to find the area of the hypotenuse square, *ABDE.* Does this support the Pythagorean theorem?

5. Drag the vertices of the triangle. You can change its dimensions, but it will always be a right triangle.

Q6 Change the leg dimensions to $a = 4$, $b = 7$. Using the same procedure as above, show that this triangle supports the Pythagorean theorem too.

THE DISTANCE FORMULA

In the steps below you will use the Pythagorean theorem to find the distance between two specific points from their coordinates. You will finish by writing a general formula for the distance between any two points on the coordinate plane.

6. Go to page 2. This page contains points *A* and *B* on the coordinate grid.
7. Press the *Show Triangle* button. This shows you right triangle *ABC,* like the triangle in the previous section but with the hypotenuse labeled *d* for "distance."

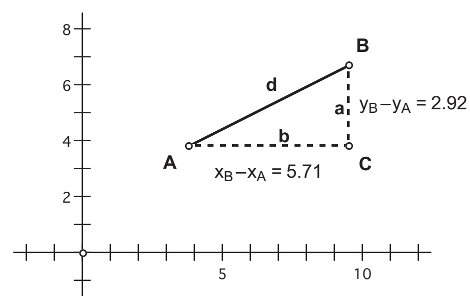
Triangle *ABC* is a right triangle, so the Pythagorean theorem should apply.

$$d^2 = a^2 + b^2$$
$$d = \sqrt{a^2 + b^2}$$

8. Select points *A* and *B*. Choose **Measure | Abscissa (x).** Select *A* and *B* again, and choose **Measure | Ordinate (y).**

Q7 You don't have to measure the coordinates of point *C* separately. What are the coordinates of C in terms of x_A, x_B, y_A, and y_B?

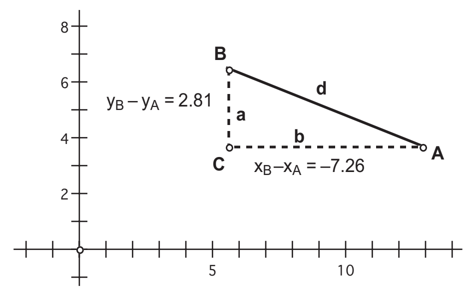
Q8 In terms of the coordinates, what are the lengths of sides *a* and *b*?



To enter a coordinate into the Calculator, click the coordinate measurement in the sketch.

9. Choose **Measure | Calculate.** Calculate sides *a* and *b* from the coordinates.
10. Choose **Measure | Calculate.** Using your calculations for *a* and *b,* use the Pythagorean theorem to calculate distance *d.*

Q9 Drag point *B* so that it is left of *A*. What happens to the calculated distance *b*? Explain why this does not affect the final distance calculation.



11. Select *A* and *B*. Choose **Measure | Coordinate Distance.** Compare this distance to the one you calculated.

Q10 Drag points *A* and *B* to different locations on the coordinate plane. Does your calculated distance always match the measurement from step 11? Are there any special conditions under which the distance formula does not apply?

The label of your distance calculation should match the distance formula.

Q11 Write down a formula for the distance *d* in terms of the coordinates of *A* and *B*. This is the *distance formula.*

Objective: Students verify the Pythagorean theorem by examining right triangles plotted on a square grid. They then apply the theorem on the coordinate grid and develop the distance formula.

Student Audience: Algebra 1, Geometry

Prerequisites: Students should be familiar with the Pythagorean theorem, but not necessarily the proof.

Sketchpad Level: Intermediate. Students perform several calculations with little guidance.

Activity Time: 30–40 minutes. The activity is in two sections, either of which can stand alone.

Setting: Paired/Individual Activity (use **Pythagorean Theorem.gsp**) or Whole-Class Presentation (use **Pythagorean Theorem Present.gsp**)

COMPARING SQUARES

Q1 $a = 2$, $b = 3$, and $a^2 + b^2 = 2^2 + 3^2 = 13$

Q2 The area of a square is the square of the length of a side.

Q3 The area of square *CFGH* is $5^2 = 25$.

Q4 Each triangle has an area of 3. Their combined area is 12.

Q5 If you remove the four triangles from the circumscribed square, what remains is the hypotenuse square, having area c^2.

$$c^2 = 25 - 12 = 13$$

This matches $a^2 + b^2$ from Q1, so $a^2 + b^2 = c^2$. This supports the theorem.

Q6 When $a = 4$ and $b = 7$, $a^2 + b^2 = 65$.

The area of the circumscribed square is $11^2 = 121$.

Each triangle has area 14, so their combined area is 56.

The area of the hypotenuse square is $121 - 56 = 65$, so again, $a^2 + b^2 = c^2$.

THE DISTANCE FORMULA

Q7 The coordinates of point *C* are (x_B, y_A).

Q8 $a = x_B - x_A$ and $b = y_B - y_A$

Q9 When *B* is to the left of *A*, the calculation for side *b* is negative. It has the correct magnitude, but the sign is negative. In the distance formula, the calculation is squared. The end result will be correct because $(-b)^2 = b^2$. The same applies to the calculation for *a*.

Q10 The calculation will match the measurement no matter what the positions of *A* and *B* are. This is one math formula with no special cases or exceptions.

Q11 $d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

WHOLE-CLASS PRESENTATION

Comparing Squares

1. Open **Pythagorean Theorem Present.gsp.** Drag points *A* and *B* around, and let the class see that the triangle vertices always fall on grid intersections.

Q1 Ask someone to explain what the Pythagorean theorem means as applied to this triangle. The legs of the triangle are *a* and *b*, and the hypotenuse is *c*, so $a^2 + b^2 = c^2$. Press the *Show Pythagorean Theorem* button.

2. Use small numbers the first time. Drag the vertices into a position such that $a = 3$ and $b = 2$.

3. Press the *Show Leg Squares* and *Show Hypotenuse Square* buttons.

Q2 Ask what the square areas represent. They represent the squares of the three triangle sides.

Q3 Show that you can substitute $a^2 = 4$ and $b^2 = 9$ into the equation. Challenge the class to find the area of the hypotenuse square without using the Pythagorean theorem.

4. Press the *Show Circumscribed Square* button.

Q4 Ask for the area of the large square (25).

Q5 Ask for the areas of the four triangles. Each of them has an area of 3.

Q6 Now ask again for the area of the hypotenuse square. Students should see that you can get this by subtracting the four triangle areas from the circumscribed square area: $25 - 4(3) = 13$

5. Make this substitution in the equation, showing that the Pythagorean theorem works in this one case. Drag the vertex points and try at least one other case.

Distance Formula

6. Go to page 2. You will see points *A* and *B* on the coordinate grid. Tell the class that the objective is to calculate the coordinate distance between the points using only their coordinates.
7. Press the *Show Coordinates* button.

Q7 Press the *Show Triangle* button. Ask the class for the coordinates of point *C*. The coordinates are (x_B, y_A).

Q8 Since the legs of the right triangle are vertical and horizontal, it is a simple matter to find their lengths by subtracting coordinates. Ask for the formulas: $a = y_B - y_A$, $b = x_B - x_A$.

8. Press the *Show Calculations for a and b* button.

Q9 At this point, you have the lengths of the legs of the right triangle. You can use the Pythagorean theorem to compute distance *d:*

$$d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

9. Press the *Show Calculation for d* button.
10. Select points *A* and *B*. Choose **Measure | Coordinate Distance.** Compare this distance to the one you calculated. You can explain to the class that this measurement is actually found using the same formula.

Q10 Drag points *A* and *B* around the screen to confirm that the calculation and the distance always concur. Depending on the positions of the points, either of the calculations for *a* and *b* can be negative. That cannot be right, since these numbers represent segment lengths. Ask for a good explanation of why the formula works even when these signs are wrong. (It is because the formula squares both of these calculations, making their signs irrelevant.)