4

Solving Equations and Inequalities

Approximating Solutions to Equations

Before you try to find an exact solution to a problem, you may find it helpful to first approximate a solution. In real life, approximations may be good enough. For instance, if you are driving to Yellowstone National Park, you may be glad to know that you will be there in approximately $3\frac{1}{2}$ hours. It may not be possible to know *exactly* how long it will take you, but knowing the approximate time will help you plan.

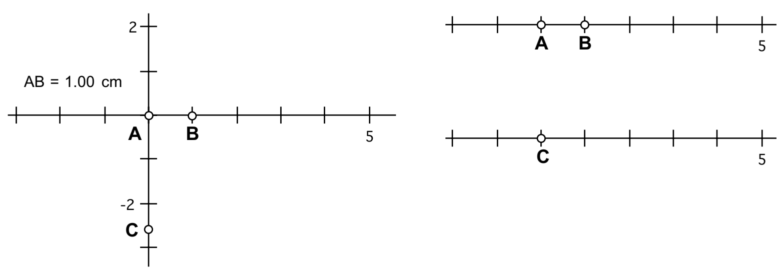
SKETCH

Begin by creating two number lines. You will do this by creating two sets of coordinate axes, and then hiding the y-axis.

1. In a new sketch, choose **Graph | Define Coordinate System.** Then choose **Graph | Hide Grid.**

Click the **Text** tool on a point to show its label, or select the point and choose **Display | Show Label.**

2. There are two points shown on the screen, one at the origin and another one defining the unit distance on the x-axis. Show the point labels. The labels should be A and B. Select both points and choose **Measure | Distance.**
3. Construct a point C on the negative y-axis. Hide the y-axis.
4. To create the second number line, select point C and distance measurement AB. Choose **Graph | Define Unit Distance.** A message will appear warning you that you are creating a second coordinate system. Click Yes. Hide the new grid and the new y-axis.



Q1 You should now have two number lines. What happens when you drag the unit point B on the upper number line?

To change a label, double-click it with the **Text** tool.

5. Construct a point on the upper number line. Select the point and choose **Measure | Abscissa (x).** Change the measurement label to x, and change the label of the new point to x also.

The point and measurement correspond to the variable x. You will use them to build both sides of this equation:

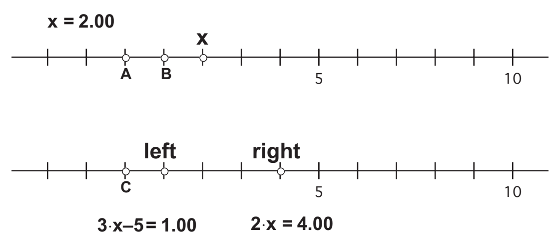
$$3x - 5 = 2x$$

When you need to enter *x* into the expression, click on the measurement *x* in the sketch.

6. Choose **Measure | Calculate.** Compute the value of the expression $3x - 5$. Then repeat the process to calculate the value of the expression $2x$. Arrange these calculations side-by-side, with the first on the left and the second on the right.
7. Choose **Graph | New Parameter.** Change the name to *zero,* and change the value to 0.
8. Select in order the left calculation ($3x - 5$) and the parameter *zero.* Choose **Graph | Plot As (x,y).**

Q2 This last step created a new point on the lower number line. If it is not in view, drag point *x* across the screen until you see it. What does the number line value of the new point represent?

9. Change the label of the new point to *left.* Use the procedure from step 8 (but using the calculation $2x$) to create another point on the lower line. This one represents the right side of the equation. Label it *right.*



Q3 Drag point *x* until you see points *left* and *right* come together. When the points meet, what is the value of *x*? What is the solution to the equation?

OTHER EQUATIONS

You can edit the calculations in order to solve other equations. Double-click a calculation with the **Arrow** tool to get back to the Calculator window. After changing a calculation, you may need to adjust the scale to see the corresponding point.

Q4 Solve these equations:

a. $31 - 3x = 2(x - 26)$ b. $\dfrac{15(x-3)}{4} = -40 - 7x$

Q5 Change the equations so that no matter where you drag *x,* there is no solution.

Q6 Change the equations so that no matter where you drag *x, left* and *right* are always equal.

Q7 So far, all of the equations have been linear, but this approximation method can work with any equation with one variable. Approximate the solutions to these:

a. $x^2 + 5x + 3 = 2x + 13$ b. $3\sqrt{x} = 2x - 6$

Approximating Solutions to Equations — Activity Notes

Objective: Students plot points on a number line to represent the left and right sides of an equation. They then drag a point to change the value of x and approximate the solution to the equation.

Student Audience: Algebra 1

Prerequisites: Students should understand what it means to solve an equation with one variable.

Sketchpad Level: Intermediate. Students construct the sketch from scratch, using detailed directions. They also use the Calculator to build expressions.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (start with a new sketch or use **Approximating Solutions.gsp,** which has steps 1–4 already constructed; you can save time by having students open this sketch and start with Q1) or Whole-Class Presentation (use **Approximating Solutions Present.gsp**)

This activity anticipates Dynagraph activities, in which independent and dependent variables are graphed on parallel number lines.

SKETCH

Q1 If the construction is correct, the unit point B on the upper number line should control the scale of both lines. Confirm that this is working before moving on.

Q2 The number line value of the new point should match the calculation on the left, $3x - 5$.

Q3 When the points *left* and *right* come together, the left and right sides of the equation are equal for that value of x. This happens when $x = 5$, and that is the solution to the equation.

Some students may make the mistake of saying that 10 is the solution, because that is what appears on both sides. Check for understanding.

OTHER EQUATIONS

Q4 a. $x = 16.6$ b. $x \approx –2.67$

The precision of students' answers will depend on the scale they choose.

Q5 Answers will vary. An equation like $x = x + 1$ works.

Q6 Answers will vary. An equation like $2x = x + x$ works.

Q7 a. $x = -5$ or 2 b. $x \approx 6.96$

The equation in part a is a quadratic equation with two solutions. If this is something new for students, ask them to explain why there are two answers. For the equation in part b, ask why the left-side calculation is undefined when x is dragged to the negative end of the number line.

VARIATIONS

You may wish to challenge students with other equations. Even if the solution is near zero, the expressions on each side of the equation may have very large magnitudes, making this model unwieldy. It is not actually necessary to see points *left* and *right* since you can see the calculations they represent. You can make this easier by subtracting one calculation from the other and finding the value of x for which the difference is zero.

WHOLE-CLASS PRESENTATION

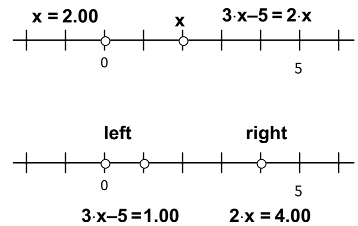
In this presentation students will observe how dragging a point on a number line can quickly generate many values to substitute in a simple equation, making it easy to find an approximate solution to the equation.

There are different ways to solve an algebraic equation in one variable: by algebraic manipulation, by graphing each side of the equation as a function and finding the point of intersection, and by substituting different values for the unknown to get closer to the answer. In this activity you will be demonstrating the last of these, by dragging a point along a number line to change the value of x continuously. This makes it easy to find approximate solutions quickly, even for equations that are quite difficult or impossible to solve analytically.

You will use the first page of the presentation sketch to solve the equation $3x - 5 = 2x$. The values of the expressions on the left and right sides of the equation correspond to two points labeled *left* and *right*. As you move x along the top number line, points *left* and *right* move accordingly along the bottom number line. Students look for the value of x that makes *left* and *right* coincide.

To present this activity to the entire class, follow the Presenter Notes and use the sketch **Approximating Solutions Present.gsp.**

Use these steps and questions to present this topic to the class.



1. Open **Approximating Solutions Present.gsp** and use it to introduce the presentation.
2. Go to page 2. Press the *Show Equation* button.

Define the term *substitute* if students don't already know it.

3. Explain that solving an equation means finding the value of *x* that makes the two sides of the equation equal.
4. Press the *Show x* button. Drag point *x* left and right to show how the value changes. Emphasize that *x* is a variable.
5. Press the *Show Left Side* button to show a point labeled *left*, corresponding to the value of the left side of the equation.
6. Drag *x* again so students can see how changing *x* changes the position and value of point *left*.

Have several students answer the questions in their own words.

Q1 Ask students how dragging *x* relates to trying different numbers for *x*. (One of their observations should be that dragging makes it easy to try a lot of numbers for *x* very quickly.)

Q2 Leave *x* someplace other than 2, and ask students if they can tell you where point *left* will go if you move point *x* to 2. (Answer: $3x - 5 = 3 \cdot (2) - 5 = 1$.)

7. Press the *Show Right Side* button, and drag *x* again. Avoid emphasizing the solution for the time being, and leave *x* at a value for which the equation is false.

Consider calculating the difference between the left- and right-side values. Students should predict that *left* − *right* = 0 at the solution.

Q3 Ask students whether the equation is now true or false. Then ask what they would expect to see if the equation were true. (Students should expect two things: The values will be equal, and points *left* and *right* will coincide.)

8. Drag *x* left and right. Ask whether each direction moves you closer to an answer or farther away. Ask if *left* will always be to the left of *right*. Drag *x* again until the points coincide.

Q4 Have students verify that this value of *x* does indeed give an approximate solution to the equation.

After changing the equation, you may need to change the scale to see the solution.

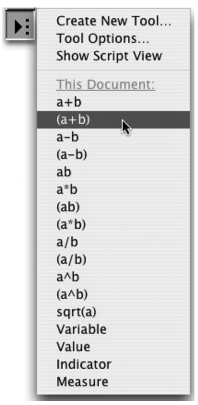
Use the remaining pages of the sketch to try different equations. Alternatively, double-click the existing calculations on page 2 to change the left and right expressions to solve any equation.

Undoing Operations

Most algebraic operations have *inverses*—operations that undo the effect of the given operation. For instance, if you add 7 to a number, you can get your original number back by subtracting 7. In this activity you will construct *algebars* that model algebraic expressions and then construct additional bars to undo the algebra and get back to the original value.

INTRODUCING THE ALGEBAR TOOLS

1. Open **Undoing.gsp.** This sketch shows a bar representing the value of x and has tools to create algebraic expressions using x.
2. Drag the red point at the tip of the x bar left and right. Observe how it behaves.



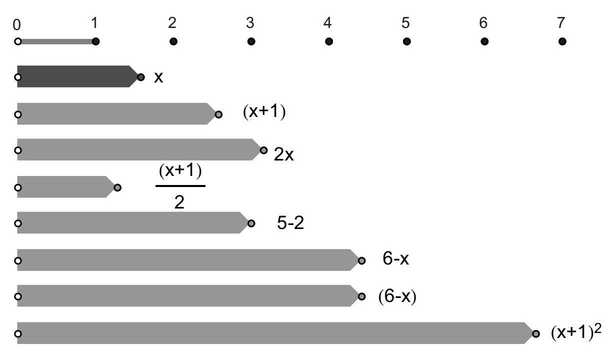
3. Just below the x bar, create a new bar for the expression $(x + 1)$: Press and hold the **Custom** tools icon, and choose the **(a+b)** tool from the menu that appears. Then click the tool on five objects: the first white point below the red bar, the red point at the tip of the x bar, the caption "x" that labels the bar, the blue point beneath the number 1, and the number 1 itself.

Q1 Choose the **Arrow** tool and drag x back and forth again. How does the $(x + 1)$ bar behave when you drag x?

Every tool with both a and b in its name requires five clicks: one for the starting point, two for the value and caption of a, and two for the value and caption of b. For values, you can use either the point at the tip of an existing bar or one of the blue points representing constants.

As you work, drag x occasionally to see how the bars behave as x changes.

4. Try out several of the tools to get a feel for how they work. Create the algebars shown below, and then make more bars of your own.



Q2 What's the difference between the **a–b** tool and the **(a–b)** tool?

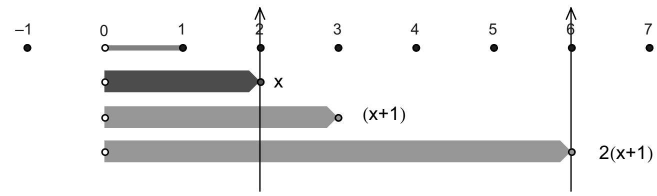
Q3 Click the **Indicator** tool on the red point at the tip of the x bar. Then click the **Measure** tool on the red point and on the caption. What happens?

CREATING AND UNDOING AN EXPRESSION

Now you'll use these tools to model the expression $2(x + 1)$, and then you'll undo the result to get back to x.

5. Go to page 2. Use the **(a+b)** tool and then the **ab** tool to build the expression $2(x + 1)$. Put an indicator through x, and another through $2(x + 1)$.

Q4 If you substitute 2 for x, what should be the value of $2(x + 1)$? First find the answer by dragging x so its value is 2. Then check by substituting 2 into the expression and evaluating it on paper. Show your work.



Q5 Adjust x so that $2(x + 1) = 3$. What value of x makes this happen?

Q6 Adjust x so that the x algebar and the $2(x + 1)$ algebar are the same size. What value of x makes this happen? What equation does this represent?

Q7 Use the algebars to solve these two equations: $2(x + 1) = 0$ and $2(x + 1) = 7$.

To get $2(x + 1)$, you started with x and performed two algebraic operations. Now you will perform the opposite operations to get back to the original value of x.

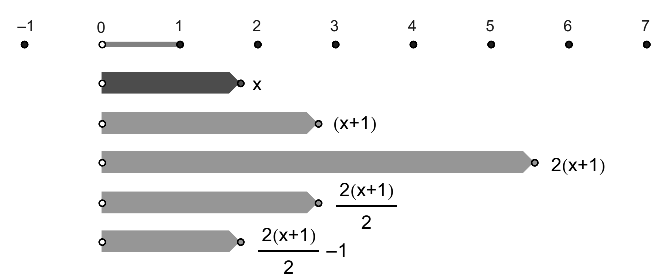
6. Use division to undo the multiplication operation: Divide the $2(x + 1)$ algebar by 2 by using the **a/b** tool.

Q8 What other algebar does this new bar match?

For step 7 you can either subtract 1 or add -1.

7. Use subtraction to undo the addition step.

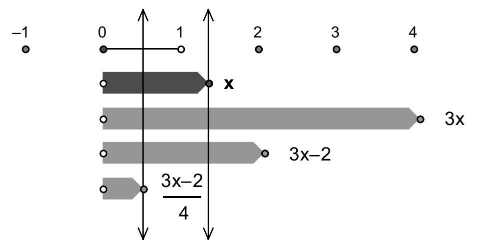
Q9 Drag x back and forth and observe the pattern made by your five algebars. Describe the shape or symmetry that they show.



ANOTHER EXPRESSION

8. Use page 3 to build this expression:

$$\frac{3x - 2}{4}$$



It took three operations to build the expression, so it takes three operations to undo it.

9. Add three more algebars to undo the expression and get back to the original value of x.

10. Put indicators on the points at the tips of the algebars for x and for $\frac{3x-2}{4}$.

Q10 Drag x so that its length matches the $\frac{3x-2}{4}$ algebar. What is the value of x, and what equation does this arrangement represent?

Q11 Drag x until it's exactly at 2. At what value is the $\frac{3x-2}{4}$ bar?

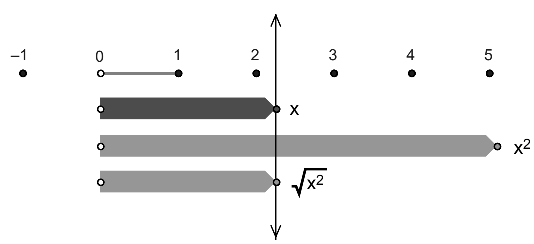
Q12 Drag x to make the value of $\frac{3x-2}{4}$ as close to 4.0 as you can make it. What is the value of x? What equation did you just solve?

EXPLORE MORE

Q13 Using the remaining pages, build each of the following expressions and then undo your operations to get back to the original value of x. Be sure to drag x to test your results.

a. $3\left(\frac{x}{2}\right) - 1$ b. $\frac{2(x-1)}{3}$ c. $2(x-3)+4$

Q14 Build the expression x^2 and then use the **sqrt(a)** tool to undo it. What difficulty do you encounter? Describe the problem you see when you drag x, and explain why the problem occurs.



Objective: Students create algebraic operations using a model that shows an image of each step, use inverse operations to undo the original operations, and observe the symmetry of the resulting pattern.

Student Audience: Algebra 1

Prerequisites: None

Sketchpad Level: Challenging. Students use custom tools.

Activity Time: 30–40 minutes. The introductory part of this activity will go more quickly if students have used the algebar tools in the Equivalent Expressions activity.

Setting: Paired/Individual Activity (use **Undoing.gsp**) or Whole-Class Presentation (use **Undoing Present.gsp**)

This activity is useful preparation for the activity Solving Linear Equations by Undoing. It makes the concept of undoing operations concrete and visual, and it provides a good introduction to the use of the algebar tools.

INTRODUCING THE ALGEBAR TOOLS

Q1 When you drag x, the tip of the $(x + 1)$ bar stays one unit to the right of the tip of the x bar.

Q2 Each tool creates a new bar showing the value of $b - a$. One tool creates a caption with parentheses, and the other creates a caption without them.

Q3 The **Indicator** tool shows a vertical line through any point so you can compare values easily. The **Measure** tool shows the numeric value of an algebar.

Q4 If you substitute 2 for x, the result is 6. Students should show work.

Q5 You must drag x to 0.5 to make $2(x + 1) = 3$.

Q6 To make the x bar and the $2(x + 1)$ bar the same length, x must be approximately -2. This represents the solution to the equation $x = 2(x + 1)$.

Q7 For $2(x + 1) = 0$, you must drag x to -1. For $2(x + 1) = 7$, you must drag x to 2.5.

Q8 The new bar, labeled $\frac{2(x+1)}{2}$, matches the $(x + 1)$ bar.

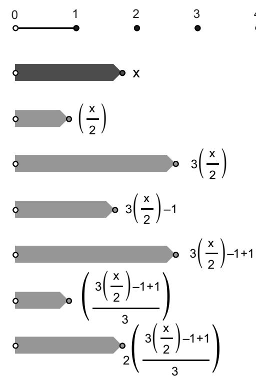
Q9 The bottom bars are a reflected image of the top bars, with the middle bar serving as the mirror.

Q10 When $x = \frac{3x-2}{4}$, the value of x is –2.

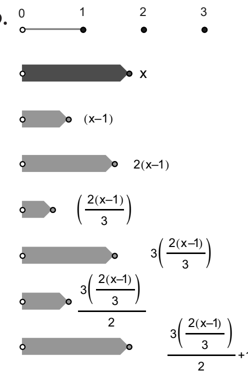
Q11 When x is exactly 2, the value of $\frac{3x-2}{4}$ is 1.

Q12 The value of x is 6. The equation is $\frac{3x-2}{4} = 4$

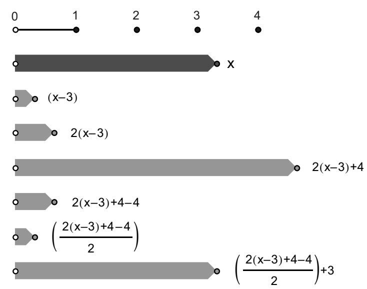
Q13 a.



b.



c.



Q14 If you make x negative, the bars no longer match. Squaring a number cannot be undone uniquely. Of the two possible results for undoing this operation, the **sqrt(a)** tool produces only the positive one.

WHOLE-CLASS PRESENTATION

In this presentation students view a model of algebraic variables and expressions in which each value is represented by the length of a bar on the screen. Students see how dragging x makes it easy to substitute values in the expression and to find trial-and-error solutions to equations. They also see how undoing algebraic operations results in a distinctive visual pattern.

The mechanics of using the algebars tools to build the expressions are important for a hands-on student activity, but not for a whole-class presentation. To avoid these mechanics, the presentation sketch (**Undoing Present.gsp**) skips the introduction of the tools and proceeds immediately to the process of creating and undoing constructions, providing buttons to avoid construction steps.

Most of the questions in this activity involve dragging x. As you present, drag x frequently to emphasize to students what it really means for x to be a *variable.*

Solving Linear Equations by Balancing

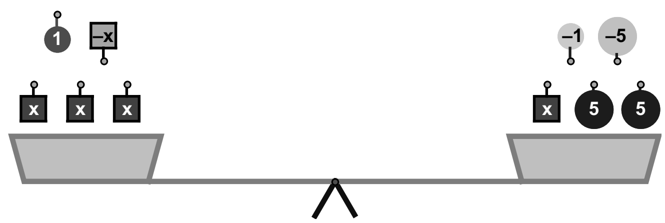
To solve a complicated equation, you can make it simpler while keeping the two sides of the equation equal. In this activity you will use a Sketchpad balance to show an equation and solve it by using operations that keep the sides balanced.

EXPLORE THE BALANCE

1. Open **Solve by Balancing.gsp.** Experiment by dragging objects from the storage area (on the left of the dividing bar) to the left or right balance pan.

Q1 Which objects weigh a pan down, and which ones pull it up?

Q2 Find a combination of weights and balloons different from the one shown below that balances the two pans. List the objects you put on each pan.



Q3 Write down the algebraic equation that corresponds to your arrangement of weights and balloons. Press the *Show Formula* button to check your answer.

When the pans are balanced, some operations disturb the balance and others do not.

Q4 Try each of these operations and write down how it affects the balance. Before each operation, make sure the pans are balanced and that each pan contains enough items to carry out that operation.

In your answer, state whether the operation makes the left pan heavier, makes the right pan heavier, or leaves the two pans in balance.

a. Drag a 1 from the storage area onto each pan.

b. Drag a 1 from the right pan to the left pan.

c. Drag a 1 and a -1 together from the left pan to the right pan.

d. Drag a 1 from the storage area onto the left pan and a -1 onto the right pan.

e. Drag a -5 onto each pan.

f. Remove an x from each pan by dragging to the storage area.

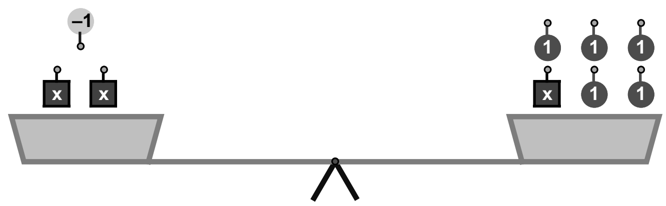
g. From the storage area drag an x onto the left pan and a 5 onto the right pan.

h. Drag an x and a $-x$ from the right pan to the storage area.

Q5 Write down at least two rules to describe things you can do that will keep the pans balanced. For each rule, write down which parts of Q4 illustrate the rule.

SOLVE AN EQUATION

Q6 Go to page 2. What equation does this balance represent? Press the *Show Formula* button to check your answer.



In the next few steps you will use these balancing rules:

Rule 1: You can drag the same kind of object from the storage area to each of the pans. (For example, you can drag a 5 to the left pan and a 5 to the right pan.)

Rule 2: If you have matching positive and negative objects on the same pan, you can remove them to the storage area. (For instance, if the left pan has both a 5 and a -5, you can remove them both.)

Make sure the pans stay in balance after each operation.

Q7 Drag a $-x$ from the storage area onto each pan. Which rule allows you to do this?

Q8 Any time you find both an x and a $-x$ on the same pan, you can remove them to the storage area. Which rule is this? How many such combinations can you find? Remove them now.

Q9 What is the resulting equation?

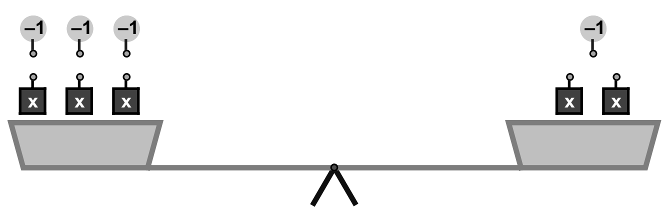
Q10 Drag a 1 from the storage area onto each pan. Any time you find both a 1 and a -1 on the same pan, you can remove them to the storage area. How many such combinations can you find and remove?

Q11 What is the resulting value of x? Press the *Show x* button to check your result.

MORE EQUATIONS

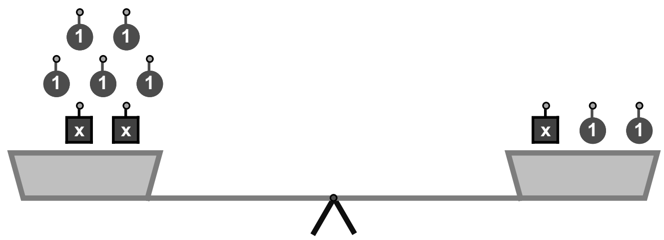
You have solved the equation when you have x by itself on one pan and only numbers on the other pan.

2. On page 3, use the rules about moving objects to eliminate as many objects as you can and to leave the last x all alone on its pan.



Q12 Write down each step that you follow, and write down the equation for the balance after each step.

Q13 On page 4, what equation does the balance show?



Q14 Use the two rules to add and remove objects until you get x on a pan by itself, with only numbers on the other pan. Each time you use Rule 1, write down what you did and the resulting equation. What is x?

Q15 Remove all the objects from the pans, and then move a single x onto the left pan. Does this object weigh the pan down or pull it up? Is your answer the same as it was for Q1? If not, why not?

Q16 Go to page 5 and add objects to the pans to model the equation $4x - 2 - x = x + 3 + x$. Then solve the equation by following the two rules.

EXPLORE MORE

Q17 There's one more rule you can use with the algebra balance. Go to page 6 and notice that each pan has two identical piles of objects. Remove one pile from the left pan and one pile from the right pan. Do the pans stay in balance? What mathematical operation did you perform on each pan?

Q18 With the help of this third rule, solve these three equations:

a. $2x - 1 = 5$ b. $4x + 3 = 2x - 3$ c. $4x + 5 = x - 7$

Q19 Page 7 contains several buttons to create different arrangements of objects. For each button, press the button and write down the equation that results. Then use the various rules you've learned to rearrange the objects and solve the equation, and write down the solution that you find.

Q20 Page 8 is a blank page. Show the value of x, adjust the slider, and arrange objects to make a balanced equation of your own choosing. Then hide the x value and slider, and challenge a friend to solve your equation.

Use Sketchpad's Help menu to find out how to make and use movement buttons.

Q21 Page 9 is also a blank page. Make a movement button to move objects out of the storage area and onto the pans to make an equation. Then make a movement button for each step in solving the equation. Use the movement buttons to demonstrate your problem to your classmates.

Objective: Students manipulate a balance model of equations, develop rules for adding to both sides and for removing zeros, and use the model to solve equations. The model works with both positive and negative weights.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None

Sketchpad Level: Easy. Students use a prepared sketch.

Activity Time: 30–40 minutes. Some of the Explore More questions could take considerably longer.

Setting: Paired/Individual Activity (use **Solve by Balancing.gsp**) or Whole-Class Presentation (use **Solve by Balancing Present.gsp**)

The balance used in this activity was made using the **Algebalance.gsp** sketch in the **Supplemental Sketches and Tools** folder. You can use these tools to create your own balance sketches; instructions accompany the sketch.

EXPLORE THE BALANCE

Q1 The positive objects (1, 5, and x) weigh the pan down, and the negative ones (-1, -5, and $-x$) pull it up. (The behavior of x and $-x$ depends on whether x is positive or negative. This is explored later in the activity, so there's no need to worry about it yet.)

Q2 Answers will vary.

Q3 Answers will vary. The equation should match the list of objects from Q2.

Q4 Students may first need to add objects to the pans to have enough objects to perform the operation.

a. The pans remain balanced.
b. The right pan goes up and the left pan goes down.
c. The pans remain balanced.
d. The left pan goes down and the right pan goes up.
e. The pans remain balanced.
f. The pans remain balanced.
g. The left pan goes up and the right pan goes down (if students have not found and adjusted the x slider).
h. The pans remain balanced.

Q5 The rules are described later in the activity. Encourage students to answer this question in their own words.

Rule 1: You can drag the same kind of object from storage to each pan. (This rule is illustrated by Q4 parts a and e. Some students may also list f, which involves removing the same object from both pans.)

Rule 2: If you have matching positive and negative objects on a pan, you can remove them to the storage area. (This rule is illustrated by Q4 parts c and h.)

SOLVE AN EQUATION

Q6 The balance on page 2 represents $2x - 1 = x + 5$.

Q7 Rule 1 allows you to drag identical objects to each pan.

Q8 Rule 2 allows you to eliminate pairs like x and $-x$. Students can use this rule twice, once on each pan.

Q9 The resulting equation is $x + (-1) = 5$. The sketch is limited in how it can display equations, and the appearance in the sketch is $x + -1 = 5$.

Q10 Students can remove a single combination of 1 and -1 from the left pan.

Q11 The resulting value of x is 6.

MORE EQUATIONS

Q12 Here are the steps, though students may change the order in some ways. Students may also combine similar steps such as b and c, or they may use step g to add three 1's to each pan.

a. Add $-x$ to both pans: $2x + (-3) = 1x + (-1)$
b. Remove x and $-x$ from the left: same equation
c. Remove x and $-x$ from the right: same equation
d. Add $-x$ to both pans: $x + (-3) = -1$
e. Remove x and $-x$ from the left: same equation
f. Remove x and $-x$ from the right: same equation
g. Add 1 to both pans: $x + (-2) = 0$
h. Remove 1 and -1 from the left: same equation
i. Remove 1 and -1 from the right: same equation
j. Add 1 to both pans: $x + (-1) = 1$
k. Remove 1 and -1 from the left: same equation
l. Remove 1 and -1 from the right: same equation

m. Add 1 to both pans: $x = 2$

n. Remove 1 and -1 from the left: same equation

o. Remove 1 and -1 from the right: same equation

The last formula ($x = 2$) shows the value x must have to make the pans balance.

Q13 On page 4, the balance shows $2x + 5 = x + 2$.

Q14 To solve this equation, here are the steps:

a. Add $-x$ to both sides: $x + 5 = 2$

b. Add -1 to both sides: $x + 4 = 1$

c. Add -1 to both sides: $x + 3 = 0$

d. Add -1 to both sides: $x + 2 = -1$

e. Add -1 to both sides: $x + 1 = -2$

f. Add -1 to both sides: $x = -3$

Students may consolidate some of steps b–f by dragging more than a single -1 at a time.

Q15 A single x on the left pan pulls it up, even though it weighed it down in Q1. In Q1 (on page 1) x was positive, causing it to weigh the balance down. Here, x is negative, so it pulls the pan up. If you drag the x slider so that x is positive, this page will give the same result as page 1. If possible, have students try this by putting a single x on the left pan and then dragging the x slider to both positive and negative values.

Q16 The solution to $4x - 2 - x = x + 3 + x$ is $x = 5$.

EXPLORE MORE

Q17 When each pan has two identical columns, removing one column from each pan will keep the pans in balance. This is equivalent to dividing the number of objects on each pan by two.

Q18 a. $x = 3$ b. $x = -3$ c. $x = -4$

Q19 $-2x + 1 = -3x + 4$: $x = 3$

$2x - 2 = 3x + 1$: $x = -3$

$-x + 3 = -2x + 1$: $x = -2$

Q20 Students will make different equations. If possible, have some students show their equations and the method to use in solving them.

Q21 It may help students to study page 7 to see how the buttons on that page work.

CLASS DISCUSSION

Some fundamental principles arise during this activity.

Rule 1 is the addition property of equality. Have students explain this rule in their own words during the discussion. If they are already familiar with this property by name, encourage them to make the connection between the concrete action of dragging objects on the screen and the abstract formulation of the property. If they are not familiar with the property by name, this may be a good opportunity to have students come up with their own name for the property that is more descriptive.

Different students will justify Rule 2 differently. A class discussion will help them develop insights into the concepts of additive identity and the additive inverses, and how they can be used to solve equations. There's no need to name these concepts; the important point is for students to realize that the sum of a number and its additive inverse is zero, so that removing the combination from one side of an equation leaves the two sides equal.

A discussion of the balance on page 4, and of Q15 in particular, provides an opportunity for students to recognize that $-x$ is not necessarily negative, any more than x is necessarily positive. This important realization can prevent many errors and misunderstandings later.

Finally, students are told several times during the activity to try to get a single x on a pan all by itself. The activity doesn't try to explain this strategy, so a class discussion is a good opportunity for students to think about this strategy and understand why it's useful.

WHOLE-CLASS PRESENTATION

In this presentation students see the effects of placing both positive and negative objects on the left and right pans of a balance, and connect the visible objects with a simplified formula corresponding to the current state of the balance. This presentation helps students develop a mental model of equations and of the techniques used to solve them.

Use **Solve by Balancing Present.gsp** and the Presenter Notes to conduct a presentation for the entire class.

Solving Linear Equations by Balancing

Introduce the presentation.

1. Open **Solve by Balancing Present.gsp** and show how the balance works. You can press the numbered action buttons to show how the balance responds to the objects, or you can just drag objects to and from the balance.

   As you press the buttons or drag the objects, explain how an equation is like a two-pan balance, with equality corresponding to the pans being in balance and inequality corresponding to one pan being heavier than another.

Illustrate two rules students can use to solve equations.

2. Use page 2 to demonstrate one of the rules that makes it possible to solve equations by balancing: the addition property of equality. Press the *Show Premises* button to show the premises and to reveal buttons that allow you to illustrate the rule.
3. Use page 3 to demonstrate a second rule: Opposites on the same side of the equation add up to 0, and you can remove them.

Use the rules to solve several equations.

4. Use page 4 to show that you can solve simple equations using these rules. Press each numbered button in turn to show the various steps in solving the equation. Discuss how the solution becomes obvious once the balance has been manipulated so that there's only a single x left, on a pan by itself.
5. Use page 5 to solve another simple equation. The value of x on this page is negative, so that an x object pulls the balance up instead of weighing it down, and a $-x$ object pulls it up. After you finish solving the equation, press the *Reset* button to clear the pans, and show what happens when you drag the x and $-x$ objects on and off the pans. Discuss this with your students, and encourage them to describe and explain this behavior in their own words.

If students are not ready to think of the operation on this page as multiplying by $\frac{1}{2}$, describe it as dividing by 2.

6. Use page 6 to suggest a third rule (the multiplication property of equality). In this example both pans have two identical columns of objects. By removing half of what's on each pan (removing half the weight on each pan), the pans remain balanced. Similarly, doubling the number of objects on each pan keeps them balanced.
7. Page 7 shows the solution of a more complicated equation using the balance.
8. Page 8 is blank. You can use it to set up and solve any of the problems from the student activity pages.

Finish with a class discussion. As part of the discussion, encourage students to ask questions about and to justify these techniques for isolating x on one side of the equation while keeping it balanced.

Solving Linear Equations by Undoing

Algebra problems are often written in the form of equations that give you certain clues about a variable without stating its value directly. One method for finding the unknown value of the variable uses inverses to undo the steps that have been performed on the original variable.

SOLVE AN EQUATION

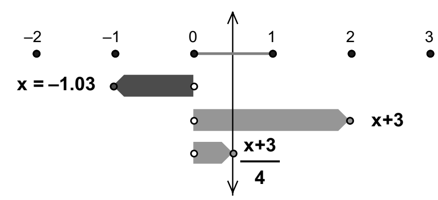
Algebars are bars that represent algebraic expressions. You can observe their behavior as the value of a variable changes.

1. Open **Solve by Undoing.gsp.** This sketch uses *algebars* to show the expression $\frac{x+3}{4}$. Drag x back and forth to observe the behavior of the different steps in the expression.

Your task is to find the solution to $\frac{x+3}{4} = 2$. You'll do it first by guess-and-check.

Press and hold the **Custom** tools icon and choose **Indicator** from the menu that appears. Click the tool on the point at the tip of the bar.

2. Use the **Indicator** custom tool to put an indicator through the point at the tip of the bottom green bar.



Q1 Drag x and observe the indicator. How does the indicator allow you to tell when you're close to the solution—that is, when $\frac{x+3}{4}$ is close to 2? Drag x until the indicator is as close to 2 as you can make it. What is the value of x?

Q2 Why is the method you just followed called "guess-and-check"?

Sometimes a guess-and-check solution is good enough. In this activity you will find exact solutions.

Now you'll use inverse operations to "undo" your way to an exact solution, starting with the final value of the expression and working your way back to the value of x.

3. According to the equation, the final value of the expression is 2. Use the **Value** custom tool to create an algebar with a length of 2. Click the tool on three objects: the first unused white point (where you want the bar to start), the blue point representing the value 2, and the caption 2 above the blue point.

The last step in creating $\frac{x+3}{4}$ was division by 4. Use the inverse of that operation (multiplication by 4) to undo the division.

Click the tool on five objects: the white point where you want to start, the blue point and caption for the 2 bar, and the blue point and caption for the number 4.

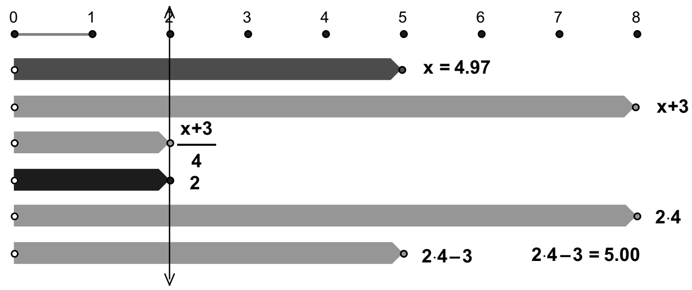
4. Use the **(a*b)** custom tool to multiply the 2 algebar by 4.

Q3 Is the resulting bar the same length as the $x + 3$ bar? Write this as an equation. Drag x back and forth. Does the bar stay the same length? Why or why not?

5. Use **Edit | Undo** to put x back where it was. (Leaving x at the approximate solution helps you to see the symmetry in the algebars.)

Q4 The first operation used to build the expression $x + 3$ was adding 3. Think about the inverse of this operation. How could you perform the inverse using addition? How could you perform the inverse using subtraction?

6. Pick one of these two methods. Choose the appropriate custom tool and use it to do the inverse operation to the 2 • 4 bar.
7. You have undone all the steps in the original expression, so this new bar shows the solution to the equation. Use the **Measure** tool to find its length.



Q5 Write down the sequence of equations for the equivalent bars, starting with the two middle bars and moving outward to the top and bottom. To get you started, the equation for the two middle bars is $\frac{x+3}{4} = 2$.

SOLVE ANOTHER EQUATION

On page 2 are algebars that you will use to solve the equation $2y - 5 = 1$.

The pattern formed by your new bars should be the reverse of the pattern of the original bars for the expression.

8. First drag y so that the bottom bar has a value close to 1. This position of y will help you see the symmetry of the bars as you construct the inverses.
9. Use the **Value** tool to create a new bar with a length of 1.
10. Use the **a+b** tool to perform the inverse of the subtraction in the expression.
11. Use the **a/b** tool to perform the inverse of the multiplication.

Q6 Use the **Measure** tool to measure your last bar. What is the solution to the equation $2y - 5 = 1$?

EXPLORE MORE

Q7 On page 3 is a more complicated equation. Use the appropriate tools to undo the operations that were performed to build it. What is the solution?

Q8 On page 4, construct this expression: $\frac{2(n-3)}{3} + 5$. Then construct an algebar with value 7, and additional algebars to solve the equation $\frac{2(n-3)}{3} + 5 = 7$.

Objective: Students use a visual model of algebraic expressions to undo the expression on one side of an equation and find the solution to the equation.

Student Audience: Algebra 1

Prerequisites: Students should be familiar with inverses, with order of operations, and with finding approximate solutions to equations by substituting (as in the activity Approximating Solutions to Equations).

Sketchpad Level: Intermediate. Students use the algebar custom tools. This activity will be easier for students who have already done activities using those tools. (Such activities include Equivalent Expressions and Undoing Operations.)

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Solve by Undoing.gsp**) or Whole-Class Presentation (use **Solve by Undoing Present.gsp**)

The reflection symmetry shown by the algebars in this activity is helpful to students conceptually; it gives a visual sense of the way in which undoing operations returns you to your starting place.

SOLVE AN EQUATION

Q1 The indicator allows you to tell when the value of the expression is close to 2. When the indicator is close to 2, the value of *x* is within a few hundredths of 5.00. (If students have not changed the scale, they cannot drag *x* to exactly 5.00. This is intentional, to emphasize the limitations of the guess-and-check process.)

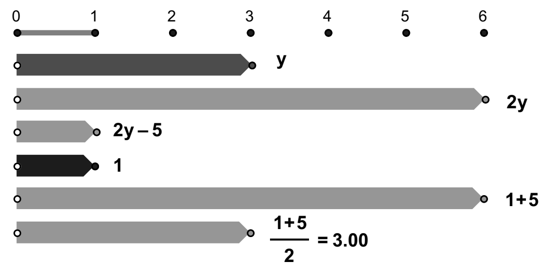
Q2 This method is called "guess-and-check" because dragging *x* to different positions is a way of guessing possible solutions and observing the position of the indicator is a way of checking how close you are to a solution.

Q3 When you drag *x* back and forth, the new bar doesn't change, but the $x + 3$ bar does. The new bar matches the $x + 3$ bar only when *x* has been dragged to make the value of the expression 2.

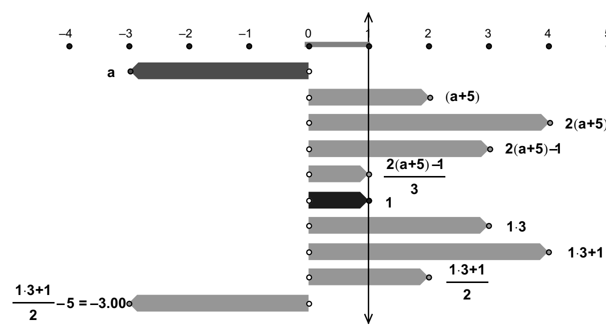
Q4 To perform the inverse of adding 3, you can either add -3 or subtract 3. Either method will work in the next step.

Q5 The sequence is $\frac{(x+3)}{4} = 2$, $(x + 3) = 2 \cdot 4$, and $x = 2 \cdot 4 - 3$. This sequence matches the algebraic steps normally used to solve the equation.

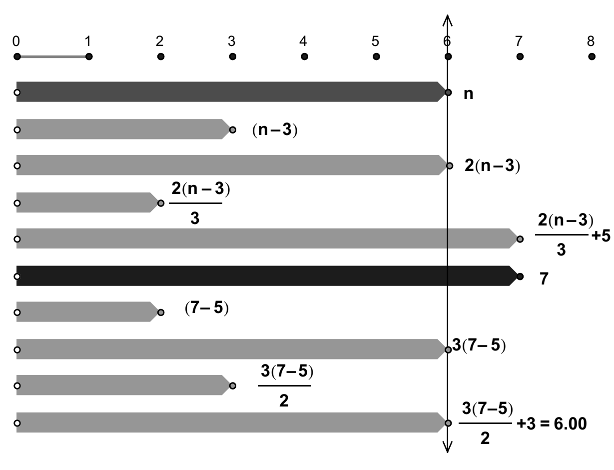
Q6 The result of measuring is $\frac{(1+5)}{2} = 3$, so $x = 3$. Here are the algebars:



Q7 The solution to the equation $\frac{2(a+5)-1}{3} = 1$ is $a = -3$. As with the earlier problems, the algebars that undo the expression must be mirror images of the ones that build the expression. Here are the algebars:



Q8 The solution is $n = 6$. Here are the algebars:



WHOLE-CLASS PRESENTATION

To present this activity to the whole class, use the Presenter Notes together with the sketch **Solve by Undoing Present.gsp.**

In this presentation students will observe a dynamic "algebar" model used to create an algebraic expression involving x and will see how performing inverse operations can undo the expression and find the original value of x.

1. Open **Solve by Undoing Present.gsp** and press the *Show Problem* button. Explain that to find x, the first step is to build the expression on the left.
2. Press the *Show x* button. Drag x left and right to show that it really is a variable. Tell students "We need to find the value of x that will make this expression equal to 2, so first we'll use x to build the expression."
3. Press *Add 3* and drag x to show how the algebar really does represent $x + 3$.
4. Press *Divide by 4* and drag x again. Ask students to stop you when x is at the right position. Encourage students to coax you to move x a bit to the left or right to make it closer.
5. Point out that it's hard to tell when you're in exactly the right spot, and press the *Show Indicator* button. Adjust x once more.

Q1 Ask students if the value of x is now exactly correct. (You have probably not been able to get closer than about 0.01 to the answer. More importantly, this is a guess-and-check solution method that may not yield an exact answer.)

6. Tell students that to get a precise answer, they must work backward from known facts, and it's a fact that the right side of the equation is 2. Press the *Show 2* button.

Q2 Ask, "To work back from the 2 to the x, the first thing we undo must be the last thing done. What operation was that?" (Students will answer, "Division by 4.")

7. Press the *Undo Division* button. Call students' attention to the fact that this new algebar, $2 \cdot 4$, is exactly the same size as the $x + 3$ bar. Write an equation.

Q3 Ask students what operation to undo next. (The answer is "Add 3.") Then ask them to give two different ways to undo the operation of adding 3. You should get responses of both "Subtract 3" and "Add -3."

Q4 Click the *Undo Addition* button and ask students which of the two methods was actually used. Write another equation.

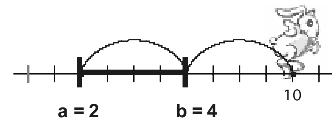
Q5 Ask students what the solution is and whether they think this solution is exact or approximate. Develop this into a discussion.

Q6 Ask students whether the bars will always form such a precise pattern. Drag x to show that the pattern holds only when x is dragged nearly equal to the solution.

8. If there's time, use the remaining pages to explore some of the problems from the student activity.

Solving Linear Equations by Jumping

In this activity you will use "rabbit races" as a way to think about and solve algebraic equations.



RABBIT RACES

1. Open **Solve by Jumping.gsp.**

Try to predict the outcome of the races in advance.

2. Press the *Race!* button to run a race. Try the other buttons to see what they do, and drag the numbers (not the bars) to change the starting positions and jump sizes. Run races with different starting positions and jump sizes.

Q1 On page 2, predict how many jumps it will take for Percy to catch up to Quincy. How did you make your prediction?

Drag the numbers themselves to change their values.

3. Check your prediction by running the race. Then press *Reset* and *Show x.* Drag the green number x to run the race again, this time in slow motion.

Q2 Reset again and set Percy's jump size to 5. Will Percy catch up to Quincy after fewer jumps or more jumps? Why?

WRITING EQUATIONS

The bottom row of the table is not fixed. The numbers in the bottom row change when the sketch changes.

4. On page 3, press the *Show Tables* button, and double-click each table to enter the current values permanently.

x	Percy
0.00	-3.00
1.00	1.00
2.00	5.00

x	Quincy
0.00	9.00
1.00	11.00
2.00	13.00

5. Run the race one jump at a time by clicking the *Jump Once* button. After each jump, double-click the tables again. Observe the values that appear in each of the tables.

Q3 What is Percy's position at the beginning of the race, when $x = 0$? By how much does his position change after each jump?

Q4 Where will Percy be after ten jumps? Without using the table, how can you quickly calculate Percy's position after any number of jumps?

Q5 Write your answer to Q4 as an equation that gives Percy's position for any number of jumps. Write another equation for Quincy's position. Use p and q to stand for the positions of the rabbits.

At the moment when Percy catches Quincy, the positions of the two rabbits are the same. You could write this in the form of an equation: $p = q$.

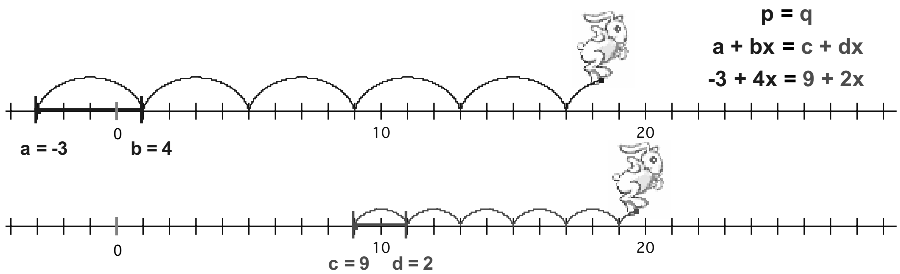
Q6 Write the equation $p = q$ in another form, using your expressions from Q5 in place of p and q.

SOLVING EQUATIONS

Solving an equation means finding the value of a variable that makes the left and right sides equal. In this section you'll find the value of x when Percy catches Quincy.

Q7 On page 4, press the *Show Equation* button. Does the equation match your answer to Q6?

Q8 Where will each rabbit be after six jumps? What does this answer indicate about the solution to $-3 + 4x = 9 + 2x$?



Q9 How far apart are the rabbits at the start of the race? How can you calculate this from a, b, c, and d in the equation?

Q10 By how much does the distance between the rabbits increase or decrease for each jump? How can you calculate this from a, b, c, and d in the equation?

Q11 How can you use your last two answers to predict when Percy catches Quincy?

Q12 Change the race so that the rabbits' positions are determined by $-3 + 5x = 9 + 3x$. Explain how you can predict that Percy will still catch Quincy in the same number of jumps. Why does the equation $-3 + 5x = 9 + 3x$ have the same solution as $-3 + 4x = 9 + 2x$?

Drag the top number line to move all three number lines left or right to see the outcome of the race.

Q13 Run this race: $9 - 5x = -3 - 3x$. Why do the rabbits both move to the left? Why does this equation still have the same solution as the one in Q12?

Q14 Set up a race to solve $21 + 4x = 3 + 7x$. What happens? How can you use your methods to solve it anyway, even though the rabbits run off the screen?

Q15 Run this race: $17 - 2x = -13 + 3x$. How is it different from the other races you have run? Explain how you can still use the same methods to solve it.

EXPLORE MORE

Q16 Imagine a race in which Percy and Quincy always stay the same distance apart. What would the two expressions for their positions look like?

Q17 Set up this race: $2x + 8 = 4x + 20$. What happens? Why? Does the equation have a solution? Try to figure out how to show the solution in your sketch.

Solving Linear Equations by Jumping

Activity Notes

Objective: Students examine the motion of two rabbits who move at constant rates, and use distances and rates to write and solve equations of the form $a + bx = c + dx$.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should be able to operate with signed numbers and think about constant rates.

Sketchpad Level: Easy. Students use a pre-made sketch.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Solve by Jumping.gsp** in either setting)

RABBIT RACES

Give students time to experiment with the sketch and run some races. See what questions students have about the motion of the rabbits. Note that the jump size and starting positions take on integer values only.

Q1 Quincy has a lead of 12 units. Since Percy moves 4 units with each jump while Quincy moves only 2, this lead diminishes by 2 units per jump. The 12-unit lead divided by 2 units/jump equals 6 jumps.

Running the race in slow motion may allow students to see how the lead diminishes. For Q1, students might also reason using guess-and-check to determine the location of each rabbit after a certain number of jumps or repeatedly add the jump sizes to the initial position.

Q2 Since Percy moves faster in this race, it takes fewer jumps to catch up. Since Percy's jump size is now 5 units, Quincy's lead diminishes by 3 units per jump. The 12-unit lead divided by 3 units/jump equals 4 jumps.

Q3 At $x = 0$, Percy is at -3 on the number line. His position increases by 4 with each jump.

Q4 After 10 jumps, Percy is at 37 on the number line. You can calculate this by adding 4 ten times to the initial position of -3. For any number of jumps, 4 times the number of jumps added to the initial position will give the current position.

Q5 The equation $p = -3 + 4x$ gives Percy's position, while $q = 9 + 2x$ gives Quincy's.

Q6 $-3 + 4x = 9 + 2x$

Q7 If students are not familiar with writing equations such as these, they can spend additional time on page 4 setting up races and making tables, and then checking to see if their equations are correct.

Q8 When $x = 6$ jumps, $p = -3 + 4x = 21$ and $q = 9 + 2x = 21$, so $x = 6$ must be a solution to the equation.

Q9 At the start they are separated by 12 units, which is the difference between c and a.

Q10 Decrease by 2, since 2 is the difference between b and d.

Q11 As in Q1, divide the 12-unit lead by 2 units per jump.

Q12 Percy will catch Quincy since the lead is still 12, and it diminishes at a rate of 2 units per jump. Algebraically, the equation has the same solution because for $x = 6$, both expressions have a value of 27.

Q13 The rabbits both move to the left since the jump sizes are negative. The equation has the same solution, however, since Percy still has a lead of 12 and the lead still diminishes at a rate of 2 units per jump.

Q14 Here, Percy has the lead. Quincy will catch up by moving at a faster rate. The solution to the equation is still $x = 6$, as the lead is 18 at the start, and diminishes by 3 for each jump made, since 7 is 3 greater than 4.

Q15 The rabbits run toward each other, starting 25 units apart. The gap between them diminishes by 5 for each jump, since Percy moves 2 to the left, and Quincy 3 to the right, with each jump. The 25-unit difference divided by 5 units per jump equals 5 jumps.

EXPLORE MORE

Q16 If the jump sizes are the same and the starting positions are different, the distance between the rabbits remains constant. The value of b and d is the same.

Q17 In such a race, Quincy starts with a lead of 12. Because Quincy's jump size is 2 greater than Percy's, this gap does not diminish—it widens. The equation has a solution of $x = -6$. To show the solution, move the green jump value to the left of zero and run the race.

In this presentation you will model the motion of two rabbits who move at constant rates, and use distances and rates to write and solve linear equations.

1. Open **Solve by Jumping.gsp.**
2. Press the *Jump Once* button to demonstrate how the race will proceed with the given values of *a, b, c,* and *d*. Press the *Reset* button, change the values, and show the first two jumps in a different race to demonstrate how the races work.

Q1 Go to page 2 and ask, "How many jumps will it take for Percy to catch up to Quincy?" Follow up with "How did you make your prediction?"

Discuss the different ideas students have, which may include adding the jump size of each rabbit repeatedly to their initial positions until the positions are equal.

3. On page 3, run the race with the *Race* button. Then press *Reset* and *Show Tables.* Use *Jump Once* to run the race one jump at a time. Ask what happens to the distance between the rabbits with each jump made.

Q2 Reset again and set Percy's jump size to 5. Ask, "Will Percy catch up to Quincy after fewer jumps or more jumps? Why?"

Ask students to make predictions for a variety of different races, each time asking them to explain how they arrived at their prediction. Focus the questions on how the initial distance between the rabbits and the difference in their jump sizes determines how many jumps it takes for Percy to catch up.

For students who are not familiar with writing expressions for linear sequences like $-3, 1, 7, 11$, spend time creating some different races and discussing the idea of how $-3 + 4 + 4 + 4 + 4 + 4$, or $-3 + 4(5)$, is the position of Percy after 5 jumps, and how $p = -3 + 4x$ can be used to determine the position after x jumps.

Once equations for the position of each rabbit are determined, allow students to note that $-3 + 4x = 9 + 2x$ is *not* a true statement for most values of x. Explain that solving the equation means finding a value of x for which the statement is true.

Use Q8–Q15 in the activity. Focus on interpreting the equations in terms of the races and on understanding why the solutions to the equations in Q12–Q14 are all the same.

Q3 Ask students to write another equation that has a solution of 6.

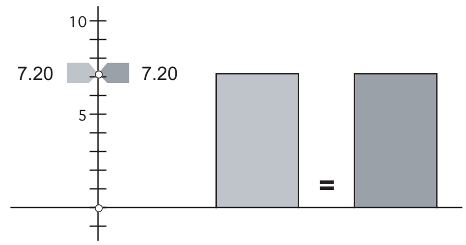
Q4 Ask students to think in terms of the race to explain why the equation $-3 + 2x = 9 + 4x$ does not have a solution of 6.

Properties of Inequality

A good general rule for equations is that if you do the same thing to both sides, they will always remain equal. This rule applies to addition, subtraction, multiplication, division, and any other operation. It would be very convenient if we could apply the same rule to inequalities. Before we jump to any conclusions, however, let's check to see how inequalities behave.

EQUALITY RULES

1. Open **Properties of Inequality.gsp.**



The two bars represent values on two sides of an equation. The numbers they represent are shown on the number line. On the left side are four buttons for the four elementary arithmetic operations.

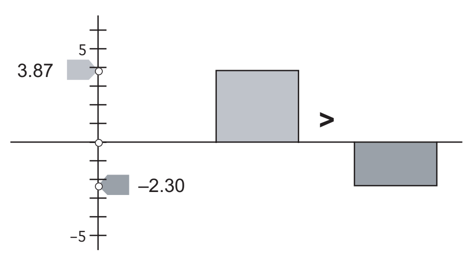
2. Press the *Add* button. This adds 2.6 to both sides of the equation.
3. Edit the value of the number next to the *Add* button. Change it to –3.8. Press the button again. This time, you are adding –3.8 to both sides.

To change the parameters next to the buttons, select a number and press the + or − key, or double-click the number to type a new value.

Q1 Experiment with all four arithmetic buttons. Try at least one positive and one negative number with each of them. Using only these buttons, is it possible to make the heights of the bars different from each other? Explain.

INEQUALITY RULES

4. On the number line, drag the red marker so that it is higher than the blue marker. Notice the change in the symbol between the two bars. Now the red value is greater than the blue value.



Q2 Once again, use all four arithmetic buttons with positive and negative numbers. Which operations can you use without changing the inequality sign?

Q3 Which operations change the inequality sign? What is a general rule to follow in these cases?

Properties of Inequality

Objective: Students use a prepared sketch to investigate arithmetic properties of inequality.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students must understand the meaning of inequality signs. This activity would be appropriate before or after they have learned about the difficulties that occur when multiplying or dividing by negatives.

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 15–20 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Properties of Inequality.gsp** for both settings)

EQUALITY RULES

Q1 Using only the buttons, it is not possible to make the heights of the bars different from each other. They are equal to begin with. For any operation, there is only one possible outcome, so they will remain equal.

INEQUALITY RULES

Q2 Addition and subtraction preserve the inequality. These operations effectively translate the ends of both bars vertically. Whichever bar is higher at the beginning must remain higher.

Multiplying or dividing by a positive number can cause the values to be closer or further apart, but the greater value will always remain greater.

Q3 Multiplying or dividing by a negative will consistently change the direction of the inequality sign. The general rule is that whenever you multiply or divide by a negative number, you must change the direction of the sign.

There is one more case that students may notice. If you multiply by zero, the inequality becomes an equality: $0 = 0$. Division by zero is not possible.

SUGGESTED EXTENSION

Ask students what happens when they square both sides of an inequality. They cannot use the sketch for this but will have to reason it out by themselves. A general rule is more difficult in this case. The side that has the greatest *magnitude* will have the greatest value after squaring. Therefore, there can be no general rule for unknown values.

WHOLE-CLASS PRESENTATION

In this presentation students see a visual model in which the same operation is applied to two different quantities that are initially equal, and observe that the quantities remain equal. When the model is changed so that the initial quantities are unequal, students observe that the sign of the inequality sometimes changes. They formulate a rule to describe when the sign changes.

To do the presentation, follow the directions on the student activity page. When investigating equality, be sure to try both positive and negative numbers for each operation. The easiest way to change one of the numbers from positive to negative may be to select the number and hold down the minus sign on the keyboard until the number changes sign. (You can press the + and − keys to increase or decrease the value of a selected parameter.)

In step 4, drag each marker in turn, both up and down, to verify that the inequality sign between the two bars is always accurate.

In Q2, when students first detect a change in the direction of the inequality, use **Edit | Undo** to revert to the previous step. Then perform the operation again. Students may find this result unexpected and want to observe the bars while the operation is performed several times. This is a good time for a class discussion in which students explain this behavior in their own words.

After students have formulated a general rule with which they are comfortable (that they must change the direction of the inequality when multiplying or dividing by a negative value), ask them what happens at the boundary between positive and negative. In other words, they haven't said what happens if the value being used is exactly zero. Ask them to make a conjecture first, before trying it in the sketch.

Similarly, ask them to make conjectures about what will happen if the two values are divided by zero, and then test the conjectures.

Solving Inequalities by Substitution

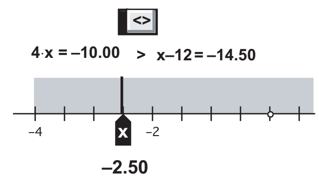
To solve an equation with one variable, you find the number(s) that can be substituted for the variable in that equation. If the substituted number makes the equation true, then it is a solution. You can solve inequalities the same way, but the solution is usually a continuous range of numbers, not just one or two.

SUBSTITUTION ON THE NUMBER LINE

A crude way to solve an inequality would be to substitute every possible number for x and see which ones satisfy the inequality. You can't do this for infinitely many real numbers. However, with a computer you can check many numbers quickly.

1. Open **Inequalities by Substitution.gsp.**

This sketch shows the inequality $4x > x - 12$. Marker x is on a number line, and its value is substituted into two calculations: $4x$ on the left and $x - 12$ on the right. Depending on the value of x, the inequality may be true or it may be false.



Q1 Drag marker x across the screen. For certain values of x, a red line segment appears above the point. What does the red line segment indicate?

To turn on tracing for the segment, select it and choose **Display | Trace Segment.**

2. Mark the interval on which x satisfies the inequality by turning on tracing for the segment and dragging the point along the number line.

Q2 What values of x satisfy the inequality $4x > x - 12$?

3. To edit a calculation, double-click it. To enter x in the Calculator, click the value of x in the sketch. To change the direction of the inequality, press the button above the inequality sign. To erase old traces, press the *Erase Traces* button.

Q3 Edit the calculations and find the solution sets to these inequalities:

a. $2x + 13 < 6x - 7$

b. $5 - 3x < 4 - 4x$

NONLINEAR INEQUALITIES

Your computer helps you do some amazing things, but it can also limit you if you depend on it too heavily. Numbers may be too big to fit on the screen, and details can be too small to notice.

To adjust the scale, drag one of the index numbers below the number line.

Q4 Try to find solutions to these nonlinear inequalities, then adjust the number line scale and see if you missed anything.

a. $240 - 20x^2 < x^3 + 76x$

b. $10000x^3 > x$

Objective: Students substitute a range of values into an inequality and determine the set of values that satisfies the inequality.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should understand what an inequality is and what it means to solve an inequality.

Sketchpad Level: Easy

Activity Time: 15–20 minutes

Setting: Paired/Individual Activity (use **Inequalities by Substitution.gsp**) or Whole-Class Presentation (use **Inequalities by Substitution Present.gsp**)

Related Activity: Solving Compound Inequalities

SUBSTITUTION ON THE NUMBER LINE

1. This inequality may be difficult to interpret at first. Each side is a calculation. In addition to the formula in each calculation, an equal sign and the numerical value also appear.

Q1 The red line segment appears if and only if x is a solution to the inequality.

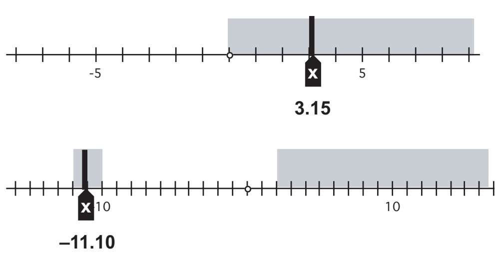
Q2 $x > -4$

Q3 a. $x > 5$ b. $x < -1$

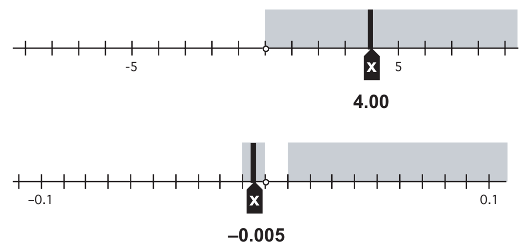
NONLINEAR INEQUALITIES

Q4 The main purpose of this section is to warn students that the computer may not be telling them the whole story. It can perform calculations for them, but it cannot think for them.

a. The range $x > 2$ is part of the solution, and students may see this by tracing it using the scale that is in use when the file is first opened. They will have to change the scale in order to see that the range $-12 < x < -10$ is also in the solution set.



b. If students trace the solution to this inequality using the original scale of the sketch, it will appear that the solution is $x > 0$. Have them change the scale so that the number line range on the screen is between -0.1 and $+0.1$. They will see that the solution is actually $-0.02 < x < -0.01$ or $x > 0.01$.



WHOLE-CLASS PRESENTATION

To present this activity to the whole class, use the sketch **Inequalities by Substitution Present.gsp.** This sketch follows the same steps as the paired/individual activity but has larger text and contains buttons to perform some of the functions.

Here are the differences:

Q1 You can use the *Animate x* button to animate x back and forth across the number line.

2. You can use the *Toggle Segment Trace* button to turn tracing on and off.

Q3 You can use pages 2 and 3 to investigate these two inequalities, with no need to use the Calculator.

Q4 Use pages 4 and 5 to investigate these two inequalities.

Finish with a class discussion. Ask students to describe what they noticed about solving inequalities that is different from solving equations and what is the same. Discuss the results of Q4; these results should help convince them that it would be nice to have a more systematic way of finding solutions, so that they won't miss the less obvious aspects of the solution.

Solving Inequalities by Balancing

Just as you solve equations by keeping the two sides of the equation balanced, you can solve inequalities by keeping the two sides of the inequality unbalanced.

EXPLORE IMBALANCE

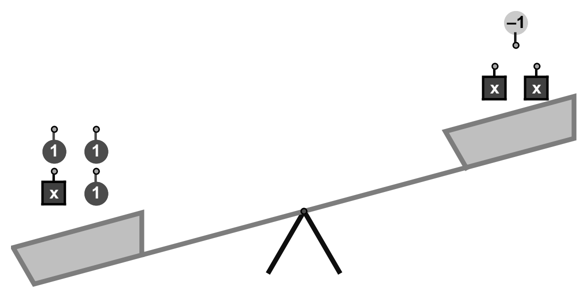
1. Open **Inequalities by Balancing.gsp.**

On page 1 the pans are balanced. Use this page to review the rules for what you can do without affecting the balance.

Q1 Drag a 1 from the storage area to each pan. Then drag a $-x$ to each pan. Do these steps disturb the balance of the pans? This is Rule 1. Write it down.

Q2 Drag an x and a -1 to each pan. Remove the x and $-x$ from the left pan. Remove the 1 and -1 from the right pan. When you remove two opposite objects from a pan, does it disturb the balance? This is Rule 2. Write it down.

Q3 On page 2 the pans are not balanced. Write down the inequality the pans represent.



As long as you follow the rules, you will not disturb the state of the pans.

Q4 Move objects on and off the pans (always following the two rules) until you get a single x all by itself on one pan and only numbers on the other pan. Write down this inequality.

Q5 Page 3 has a different arrangement, but the pans are still unbalanced. Use the two rules again to solve this inequality. Write down the original inequality, the steps you use, and the final inequality (when a single x is left on a pan by itself).

Rule 3 for inequalities is more limited than it is for equations. Be sure to use it to multiply or divide only by numbers that you know are positive.

Q6 On page 4 the pans are balanced again. This time the objects are arranged in two identical stacks on each pan. What happens if you remove one complete stack from each pan? What arithmetic operation does this correspond to? This is Rule 3. Write it down.

Q7 Page 5 has unbalanced pans. Write down the inequality. Use the rules to get a single x on one pan and only numbers on the other. Write down the solution.

EXPLORE MORE

You can adjust the value of x if you want. Press the *Show x* button and then move the slider.

Q8 Page 6 has empty pans. Create your own problem by dragging objects onto the pans. Make sure this is a problem that can be solved without fractions. Save your problem, and ask a classmate to try it.

Solving Inequalities by Balancing

Activity Notes

Objective: Students manipulate a balance model for inequalities (actually an imbalance model). They use rules that won't change the state of the balance to solve inequalities.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should be comfortable solving simple equations, and they should have already done the activity Solving Linear Equations by Balancing.

Sketchpad Level: Easy. Students manipulate a prepared sketch.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Inequalities by Balancing.gsp** for either setting)

The balance used in this activity was made using the **Algebalance.gsp** sketch in the **Supplemental Tools** folder. You can use these tools to create your own balance sketches; instructions accompany the sketch.

EXPLORE IMBALANCE

Q1 **Rule 1:** Dragging the same type of object onto each pan does not change the balance.

Q2 **Rule 2:** Removing an object and its opposite (for instance, x and $-x$, or 1 and -1) from a pan does not change the balance.

Q3 The pans represent the inequality $x + 3 > 2x - 1$.

Q4 The solution is $4 > x$.

Q5 The original inequality is $3x - 2 < 2x - 3$. To solve it, follow these steps:

a. Move two $-x$ objects onto each pan. (Rule 1)

b. Remove zeros, resulting in $x - 2 < -3$. (Rule 2)

c. Move two 1 objects onto each pan. (Rule 1)

d. Remove zeros, resulting in $x < -1$. (Rule 2)

Q6 Removing exactly half of the objects on each pan cannot change the balance. Though the example involves multiplying by one-half (or dividing by two), Rule 3 actually states that you can multiply or divide both pans by any positive number. (Negative numbers work here for equality, but not for inequality.) Students will formulate Rule 3 in different ways, some more general and some more limited. This is a good question to pursue in a class discussion.

Q7 The inequality is $x + 3 > 3x - 3$. The solution is $x < 3$.

CLASS DISCUSSION

Discuss with the class the differences between using the balance to solve equations and using it to solve inequalities.

A very important difference to note is that it's possible to violate the rules without any visible effect, depending on the nature of the violation and the value of x. For instance, if the correct solution is $x < 3$, and a student makes a mistake and ends up with $x < 4$, she will not see any change in the state of the balance, even though she now has the wrong solution set. The state of the balance gives information for only a single value of x, not for the entire solution set.

A discussion of Rule 3 is also important, both because some students will formulate it in different ways in Q6 and because the rule is different for inequality than it is for equations. Students will understand Rule 3 better if they have done the activity Properties of Inequality.

Finally, review the value of the strategy in which students try to get a single x on a pan all by itself. Understanding the value of this arrangement of the balance will help students in solving equations and inequalities symbolically.

WHOLE-CLASS PRESENTATION

Use **Inequalities by Balancing.gsp** to conduct a presentation for the entire class. Follow the steps of the student activity, and involve the class in the process of answering the questions in the activity.

Solving Compound Inequalities

A *simple inequality* declares one condition that must be true. For example, to satisfy the inequality $2x < 3 + x$, $2x$ must be less than $3 + x$. A *compound inequality* declares two or more conditions, and some combination of them must be true. For this activity, we will stick to two inequalities.

SUBSTITUTION ON THE NUMBER LINE

1. Open **Compound Inequalities.gsp.**

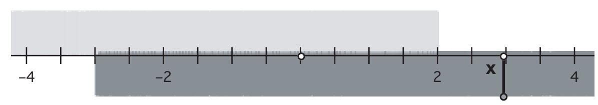
Point x is attached to the number line, and its coordinate is used to evaluate the two inequalities above the number line:

$$2x < 6 - x \qquad\qquad 7x + 10 > x - 8$$

Q1 Drag point x along the number line. A line segment sometimes appears above the points, and another one sometimes appears below. For each line segment, what determines whether it will appear?

2. Select both line segments. Choose **Display | Trace Segments.**

3. Slowly drag point x to trace the solutions of both inequalities.



Q2 What is the solution set of the left (red) inequality?

Q3 What is the solution set of the right (blue) inequality?

Q4 Consider this compound inequality:

$$2x < 6 - x \quad \text{or} \quad 7x + 10 > x - 8$$

When the word "or" is used, the solution is the *union* of the two sets.

The word "or" indicates that the solution includes all values of x for which one or both inequalities are true. What is the solution set of this compound inequality?

Q5 This is a different compound inequality:

$$2x < 6 - x \quad \text{and} \quad 7x + 10 > x - 8$$

When the word "and" is used, the solution is the *intersection* of the two sets.

The word "and" indicates that the solution includes only the values of x for which both inequalities are true. What is the solution set of this compound inequality?

Q6 Model the compound inequalities on the next page, and report their solution sets. To change the direction of an inequality sign, click the button above the sign. To edit one of the four expressions, double-click it; to enter x into the calculation, click the measurement x in the sketch. After you set the inequality signs and expressions correctly, press the *Erase Traces* button, and drag x to see the new trace.

a. $13 - x > -5x - 15$ or $4x + 10 > 17 - 3x$

b. $13 - x > -5x - 15$ and $4x + 10 > 17 - 3x$

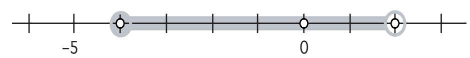
c. $30 - 13x > 30 - 7x$ or $8x + 18 < 11x$

d. $30 - 13x > 30 - 7x$ and $8x + 18 < 11x$

SYMBOLIC SOLUTIONS

The method you have been using amounts to guess-and-check. It's more reliable and often more efficient to solve both inequalities symbolically by undoing operations, and then to compare the solution sets.

4. Page 2 shows a number line plot of the compound inequalities $x \geq -4$ and $x < 2$. Notice that the inequality is already solved.



Q7 Why is the circle at -4 filled while the one at 2 is open?

5. Take a few minutes to experiment with the objects on this page. Press the buttons and edit the small numbers above the inequalities.

Q8 To solve the compound inequalities below, solve the parts separately and combine the solutions using the sketch. In each case, report the solution set and sketch a number line plot.

a. $6x < 9x - 30$ or $4x + 12 > -x + 2$

b. $5x \leq 8x + 24$ or $7x - 10 < 2x + 15$

c. $5x - 9 > 12 - 2x$ and $3x + 13 \leq 45 - x$

THE INEQUALITY GAME

The One Inequality page of **Inequality Game.gsp** shows an inequality (or sometimes an equation). Press the *Play* button to change it. Then work the graph out by yourself and press the *Show* button to check your answer.

Open the Compound Inequality page of the same document. Press the *and/or* button to choose which type of compound inequality to use.

The One Graph and Compound Graph pages provide the same games in reverse: You see the graph and must derive the expression. On the Compound Graph page, if your answer does not match the one on the screen, check again. You may be right. There is often more than one expression that will produce the same graph.

Objective: Students solve compound inequalities in one variable and plot the solution set on a number line.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should first learn to solve simple inequalities by undoing operations. They should also understand how to interpret solution sets plotted on a number line.

Sketchpad Level: Easy. Students must edit calculations.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Compound Inequalities.gsp** and **Inequality Game.gsp** in either setting)

Related Activity: Solving Inequalities by Substitution

SUBSTITUTION ON THE NUMBER LINE

Q1 The red segment appears above the number line if and only if the left (red) inequality is true for the current value of *x*. The blue segment appears below the number line if and only if the right (blue) inequality is true.

Q2 $x < 2$

Q3 $x > -3$

Q4 All real numbers are solutions to this compound inequality.

Q5 $x < 2$ and $x > -3$

This can be written in the concise form: $-3 < x < 2$.

Q6 a. $x > -7$ b. $x > 1$

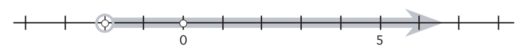
c. $x < 0$ or $x > 6$ d. No solution.

SYMBOLIC SOLUTIONS

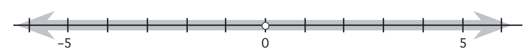
Q7 The filled circle at -4 indicates that this is at one end of the solution set, and this value satisfies the inequality. The open circle at 2 indicates that this too is at the end of the solution set, but it is not part of the solution.

5. Encourage students to experiment here. They cannot edit the numbers of the inequality directly, but they can change the smaller numbers above.

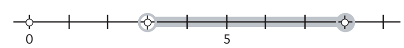
Q8 a. $x > -2$



b. All real numbers.



c. $x > 3$ and $x \leq 8$ $(3 < x \leq 8)$



THE INEQUALITY GAME

This is good practice for students who are learning these concepts for the first time. The game can continue indefinitely, and it does not require a great deal of guidance.

Each round of the inequality game requires a bit of time for calculations and thought, so it is not very fast-paced. Rather than competing against each other, it may be more effective to have students work in pairs and try to answer each challenge. Have them compare and discuss their own solutions before showing the correct answer.

WHOLE-CLASS PRESENTATION

Use **Compound Inequalities.gsp** with the Presenter Notes to present this activity to the whole class.

With compound inequalities, you may find that students' greatest problem is not solving the inequalities but correctly interpreting the and/or logical connectives. Page 1 of the Sketchpad document uses the trace feature to give a rough plot of the solutions to two inequalities. You and the class can then determine where at least one inequality is satisfied ("or") and where both are satisfied ("and").

1. Open **Compound Inequalities.gsp.** Explain to the class that the value controlled by point x is substituted into the expressions used on either side of the two inequalities.

Q1 Slowly drag point x along the number line. Tell the class to watch the red line segment above the point and the red inequality on the left. What condition determines whether the line segment will appear? (It appears if and only if x satisfies the inequality. There is a similar relationship for the blue segment that appears below.)

2. Select both of the line segments. Choose **Display | Trace Segments.** Drag point x to trace the solutions.

Q2 Derive the solutions to both inequalities symbolically in order to demonstrate that the traces match the solutions. Ask the class for the range of numbers on which at least one of the inequalities is satisfied (all real numbers), and ask them where both are satisfied ($-3 < x < 2$). Explain to them that these are the solutions to the compound inequalities below:

$$2x < 6 - x \quad \text{or} \quad 7x + 10 > x - 8$$

$$2x < 6 - x \quad \text{and} \quad 7x + 10 > x - 8$$

Remember to use the action buttons to change the directions of the inequality signs when necessary.

Q3 Edit the calculations and use the same procedure to derive the solutions to the compound inequalities below. Press the *Erase Traces* button, and drag point x.

$$13 - x > -5x - 15 \quad \text{or} \quad 4x + 10 > 17 - 3x \qquad (x > -7)$$

$$13 - x > -5x - 15 \quad \text{and} \quad 4x + 10 > 17 - 3x \qquad (x > 1)$$

$$30 - 13x > 30 - 7x \quad \text{or} \quad 8x + 18 < 11x \qquad (x < 0 \text{ or } x > 6)$$

$$30 - 13x > 30 - 7x \quad \text{and} \quad 8x + 18 < 11x \qquad (\text{No solution.})$$

Open the Compound Inequality page of **Inequality Game.gsp.** This is a bit more sophisticated. It handles six relationships ($<, >, \leq, \geq, =, \neq$) and plots the solution to a combination of two linear inequalities (or equations). Press *Play* to generate two inequalities randomly. Give students time to sketch the solution. Press *Show* to reveal the solution on the number line.