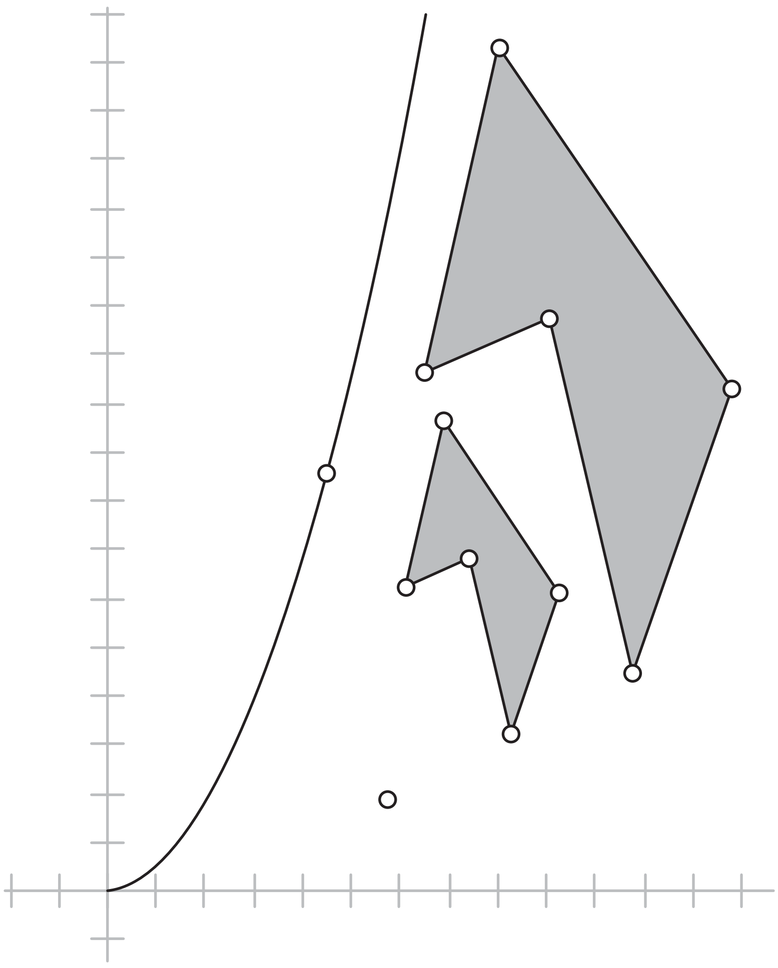
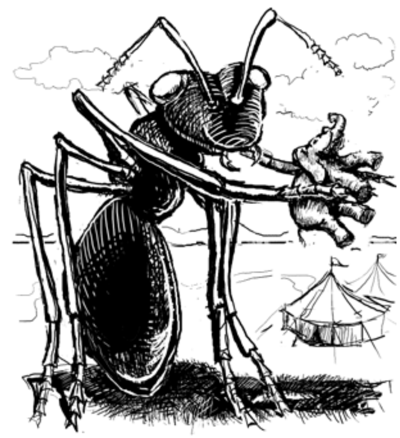
7

Quadratic Equations



Modeling with Quadratic Equations: Where Are the Giant Ants?

Some monster movies feature gigantic ants or other insects attacking and destroying cities. Did you ever wonder why such giant bugs don't exist outside the movies? Or why there aren't miniature elephants or elephant-sized ants? The answer to these questions actually relates to the evolution of species and to issues of scale. To better understand these issues, we'll scale down from three dimensions to two as you look at some "flat animals"—namely, polygons.



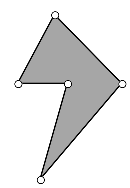
SKETCH AND INVESTIGATE

In this activity you'll explore what happens to a polygon's measurements as it gets bigger and smaller. So first you'll need to construct a polygon and its interior.

1. In a new sketch, use the **Segment** tool to draw a polygon with four, five, or six sides.

Each segment after the first should share an endpoint with the previous one. The last segment should connect back to the first. When you're done, you should have the same number of points as segments.

2. Click the **Arrow** tool in blank space to deselect all objects. Select all the points consecutively around the polygon, and choose **Construct | Interior.**

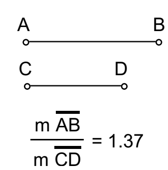


You should now have a polygon and its interior.

Click an object with the **Text** tool to show its label. Double-click the label to change it.

Next you'll make a scale copy of your interior. You'll use the ratio of the lengths of two segments as the scale factor. This will allow you to change the scale factor by dragging.

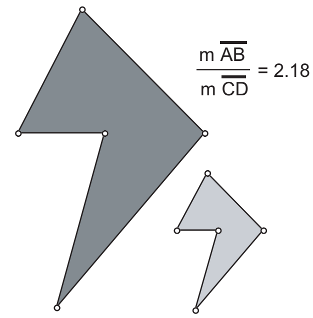
3. Construct segments AB and CD using the **Segment** tool.
4. Select in order $\overline{AB}$ and $\overline{CD}$ and choose **Measure | Ratio.** With the new ratio measurement still selected, choose **Transform | Mark Scale Factor.**



5. Use the **Point** tool to draw a point outside the polygon. With the point still selected, choose **Transform | Mark Center.**

To select the entire polygon, start in empty space and draw a selection rectangle around it with the **Arrow** tool.

6. To construct the scale copy, select the entire polygon and choose **Transform | Dilate.** Click Dilate to dilate by the marked ratio.



Dilation scales an object away from or toward a point. The scaled image appears. Drag point B to experiment with different scale factors.

Q1 What happens when $\overline{AB}$ is bigger than $\overline{CD}$? Equal? Smaller? What if $\overline{AB} = 0$? Experiment with dragging the center point and the vertices of your polygon.

7. Select a side of the scaled polygon and the corresponding side of the original polygon. Measure the ratio of these two segments.

Q2 Measure the ratio of a different pair of corresponding sides. What do you notice?

8. Measure the perimeters of the two polygon interiors by selecting both polygons and choosing **Measure | Perimeter.**

Choose **Measure | Calculate** to open Sketchpad's Calculator. Click on the perimeter measurements in the sketch to enter them into the calculation.

Q3 How do you think the ratio of these perimeters compares with the ratios you found in Q2? Why? Calculate the ratio to check your prediction.

Q4 How do the ratios of side lengths and perimeters compare with the scale factor you used for dilation? Why do you think that is?

COMPARING AREAS

You can think of perimeter as the flat ant's waist, and area as the surface area of its shell. When the ant's waist grows twice as big, what happens to the area of its shell? Think about this a moment before moving on.

Repeat step 8 and Q3, choosing **Area** instead of **Perimeter.**

9. Measure the areas of the two polygon interiors, and calculate the ratio of these two measurements.

Did you get the ratio you predicted? Drag point B until you've confirmed or refuted your prediction.

Q5 What have you discovered about the relationship between the ratio of lengths of similar figures and the ratio of their areas?

Often mathematical relationships become clearer when they are graphed. Next you'll plot the ratio of side lengths versus the ratio of areas for different scale factors.

10. Select in order the side-length ratio calculation and the area ratio calculation. Choose **Graph | Plot As (x, y).**

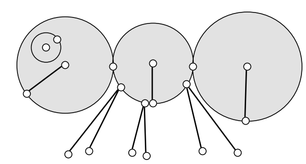
A coordinate system appears along with the plotted point. The coordinates of this point are (*side-length ratio, area ratio*) for the current scale factor.

Choose **Display | Erase Traces** if you wish to clear traces from the screen.

11. With the newly plotted point still selected, choose **Display | Trace Plotted Point.** Drag point *B* to experiment with different scale factors. Also drag the vertices of your polygon around—does anything change?

Q6 Explain why the shape of the graph makes sense given the relationship you discovered in Q4.

EXPLORE MORE



Q7 Your polygon probably didn't look much like a flat ant or elephant. For one thing, it was a little pointy. Repeat your investigation using a circle and its dilated image. Plot the ratio of radius measurements against the ratio of area measurements for the two circles. Does this change any of the relationships you found?

Select the ratios, then choose **Graph | Plot as (x,y).** Drag *B* to trace the shape of the graph.

Q8 What do you think the graph of perimeter ratio versus side-length ratio would look like? Make a prediction and then test it in Sketchpad.

Q9 Since animals are three-dimensional, let's move back to three dimensions. How does the ratio of the volumes of similar solids compare to the ratio of corresponding side lengths? Predict the answer, then go to page 2 of **Quadratic Modeling.gsp** to investigate.

Q10 The introduction to this activity discussed the sizes of animals. How do you think your results (including the Explore More questions) help to explain why ants can't be the size of elephants? (*Hint:* Animals' masses are proportional to their volumes.)

Modeling with Quadratic Equations: Where Are the Giant Ants?

Activity Notes

Objective: Students manipulate a Sketchpad model consisting of two similar polygons and explore issues of scale to better understand quadratic and linear relationships.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students need to understand the concepts of area and ratio. *Dilate* may be a new term that needs more explanation than is provided in the activity itself. Students may also need to be introduced to the term *corresponding side.*

Sketchpad Level: Challenging. This activity involves quite a bit of work in Sketchpad, with clear instructions provided for every step. You can eliminate some construction steps by starting with **Quadratic Modeling.gsp,** which shows the sketch after step 6.

Activity Time: 30–40 minutes. You can save some construction time by using **Quadratic Modeling.gsp.**

Setting: Paired/Individual Activity (use a new sketch and **Quadratic Modeling.gsp**) or Whole-Class Presentation (use **Quadratic Modeling Present.gsp**)

This activity introduces students to quadratic relationships and contrasts quadratic with linear relationships both numerically and graphically. Comparing the sizes of similar shapes will give students insight into one of the core ideas in the form and growth of animals.

Students are almost always surprised that length and area (or volume) do not grow in the same way. In order to confront their beliefs about growth, students should be encouraged to make and express their predictions prior to verifying them with Sketchpad.

Begin the discussion with a hypothetical question: Suppose ants could grow to 100 times their size (that is, 100 times as tall, 100 times as long, etc.). How would this affect their weight? How would it affect the area of one of their eyes? You can let students write down their guesses, then come back to the answer at the end of class. (The answer is that they would weigh 1,000,000 times as much, and the area of each of their eyes would be 10,000 times as large!)

This activity can provoke much class discussion and is well suited to classroom sharing. For instance, each student might choose to begin with a different type or shape of polygon. Students can also experiment with circles (see Explore More). By comparing their own results with those of classmates, they will gain a better appreciation of the quadratic growth of area for any shape. This will help them make conjectures regarding the growth of irregular shapes such as inkblots or amoebas.

Though the primary focus of this activity is length and area relationships, encourage students to investigate volume relationships. Once they have investigated area and talked about volume, they might also be able to predict what happens in higher dimensions.

SKETCH AND INVESTIGATE

6. To distinguish the new interior from the original, students can give it a different color by using the **Display | Color** submenu.

Q1 When $\overline{AB}$ is bigger, the scaled polygon is larger than the original and farther away from the dilation point. When $\overline{AB}$ is smaller, the scaled polygon is smaller and closer to the dilation point. When $AB = 0$, the image disappears.

7. Make sure students deselect all objects before selecting the two segments and measuring the ratio.

Q2 They are the same. (If they aren't, check that the sides chosen were really corresponding sides and that they were selected in the proper order—scaled polygon, then original polygon.)

8. Deselect all objects before selecting the interiors. To select an interior, click on it (and not on its sides or vertex points).

Q3 It is the same as the ratio of the corresponding sides, since perimeter is just the sum of the side lengths.

Q4 These ratios are all equal to the scale factor. Explanations will vary, but basically this is what a scale factor is: the ratio of each linear measurement in a scale drawing to that same measurement in the original.

COMPARING AREAS

Q5 Students should notice that this relationship is not linear. The ratio of the area grows faster than the ratio of the side lengths for scale factors larger than 1. To be more precise, the area ratio is the square of the side-length ratio. If the side-length ratio is 3 (3:1), the area ratio will be 9 (9:1). In terms of the "flat ant," if you double its waist size, you will quadruple the area of its skin.

10. Students may wish to hide the grid (by choosing **Graph | Hide Grid**), move the origin, and otherwise rearrange objects in the sketch to clean things up.

Q6 The graph makes sense because it appears to be a parabola (the graph of a quadratic function) and the relationship graphed is quadratic.

If students are too inexperienced with parabolas and quadratics to make this connection, they still may be able to see that when the shape is magnified, the area ratio grows faster than the side-length ratio as the scale factor gets bigger. This corresponds to the upward turn of the parabolic trace.

EXPLORE MORE

Q7 The same relationship holds with any two-dimensional shape. Students can compare circumferences or any pair of corresponding segments or distances in their figures to find that the ratio of linear measures is the same as the scale factor. The ratio of the areas of the similar shapes is the square of the scale factor.

Q8 The ratio of the perimeters of the similar shapes is the same as the ratio of the side lengths, so the graph is $y = x$.

Q9 The ratio of the volumes of similar solids is the cube of the ratio of side lengths. This is a cubic function.

Q10 An animal that is twice as long, tall, and wide as another similarly shaped animal will have four times the surface area (skin) and eight times the volume. For this reason, relatively small evolutionary size increases place great demands on the overall system. The legs of the giant ant would never be able to support the mass of its body.

WHOLE-CLASS PRESENTATION

The goal of this presentation is to use a geometric model to let students explore issues of scale. They should come away with the sense of how a scale factor affects a drawing or a model. One-dimensional and two-dimensional quantities are affected differently in a scaling. One-dimensional quantities are magnified (or shrunk) by the scale factor, while two-dimensional quantities are magnified (or shrunk) by the square of the scale factor. This applies to all shapes—circles and irregular shapes as well as polygons.

Use **Quadratic Modeling Present.gsp** to explore with the class the effects of a dilation in two dimensions. The scale factor is controlled by the lengths of the two segments $\overline{AB}$ and $\overline{CD}$. Students should watch the ratios as you drag point *B* to vary the scale factor. Changing the scale factor to a few different integer values may help students see the relationship.

Page 2 shows three-dimensional shapes. Although Sketchpad cannot truly measure the volume of three-dimensional shapes, it can calculate the volumes as the product of base area and height. The table keeps track of the various ratios (of corresponding segments, corresponding areas, and volume) for different values of the scale factor. Drag point *B* to vary the scale factor. Try a few round numbers (integer values, 0.5, etc.). Double-click the table after each one to lock in the values. Let students think about these values and how they are related.

Finish by returning to the discussion of the giant ants and having students explain in their own words why ants cannot grow to even 10 cm in length, let alone to gigantic proportions worthy of monster status.

Graphing Quadratic Functions

A quadratic function is a polynomial function of degree 2. This is its general form:

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$

Quadratic functions are not as simple as linear functions, but they do have certain predictable properties, one of which is the shape of their graphs.

TRACE A POINT

1. In a new sketch, choose **Graph | Define Coordinate System.**
2. Construct a point on the x-axis. Label it x.

To measure the x-coordinate, select point x and choose **Measure | Abscissa(x).** Double-click the **Text** tool on the measurement to change its label.

3. Measure the point's x-coordinate. Change the measurement label to x.
4. Choose **Measure | Calculate** and calculate $x^2 - 4x + 2$. To enter x, click the x measurement in the sketch.
5. Drag point x left and right. Observe the value of the calculation.
6. Select in order the measurement x and the calculation of $x^2 - 4x + 2$. Choose **Graph | Plot As (x,y).** Label the new point P.

Q1 Drag point x and watch how the height of point P changes. What is the maximum height of point P? What is its minimum height?

To trace point P, select it and choose **Display | Trace Point.**

7. Turn on tracing for point P, and trace the path of P as you drag x.

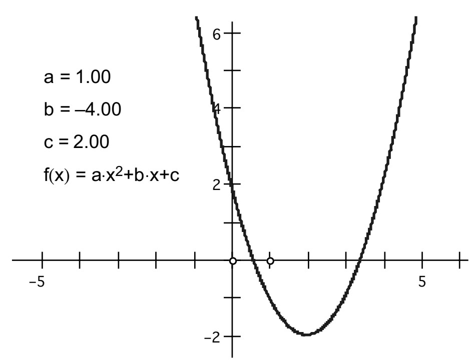
This is what graphing is all about. With numbers, you can see the value of y for only one particular value of x. With a graph, you can see the values of y for a whole set of x values.

Q2 This is the graph of $y = x^2 - 4x + 2$. Describe the shape of this graph.

GRAPH A FUNCTION

In the equation $y = ax^2 + bx + c$, the right side is a function of x. With Sketchpad, you can define a function $f(x) = ax^2 + bx + c$ and graph the equation $y = f(x)$.

8. Choose **File | Document Options | Add Page | Blank Page.**



9. On the new blank page, define a coordinate system. Choose **Graph | New Parameter** and label the new parameter *a*. Create two more parameters and label them *b* and *c*. You will change the values of the parameters later.

10. Choose **Graph | Plot New Function.** The Calculator dialog box that appears has a key labeled *x*. Enter $ax^2 + bx + c$ as the function definition. Click the parameters in the sketch to enter them.

This is a more convenient way of representing a graph in Sketchpad. You can change the parameters and the graph will change.

To make a parameter change more gradually, select it and choose **Edit | Properties | Parameter.** Change the Keyboard (+/−) Adjustments to 0.1.

Q3 Select parameter *a*. Change its value by pressing the + and − keys. What happens to the graph as you change parameter *a*? What is the shape of the graph when $a = 0$? Explain why. How does the sign of *a* influence the shape of the graph?

Q4 What happens to the graph as you change parameter *b*? What special property does the graph have when $b = 0$?

Q5 What happens to the graph as you change parameter *c*? What special property does the graph have when $c = 0$?

Q6 When you change *a* or *b*, is there always one point that does not move? What point is that? Explain why it remains fixed when all of the other points are moving.

EXPLORE MORE

Below are the coordinates of four special points on the graph. Follow the instructions to plot them on the graph.

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right); \quad (0, c); \quad \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right); \quad \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$$

11. Set the parameters back to the values they had in the first section:

$$a = 1, b = -4, c = 2$$

12. Choose **Measure | Calculate** and calculate the *x*-coordinate of the first point. Repeat for the remaining seven *x*- and *y*-coordinates.

13. Select a coordinate pair in order and choose **Measure | Plot As (x,y).** Repeat this for each of the points.

Q7 Describe the significance of each point. Vary the parameters to make sure your descriptions are accurate no matter what the shape of the graph is.

Graphing Quadratic Functions

Activity Notes

Objective: Students plot the graph of a general quadratic function and study the effects of changing the parameters.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should understand basic function notation, but they do not need any knowledge of the properties of parabolic curves.

Sketchpad Level: Intermediate

Activity Time: 20–30 minutes. The Explore More section can be omitted.

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Graphing Quadratic Present.gsp**)

Related Activity: Graphing Factored Quadratics

TRACE A POINT

Q1 Point P has no maximum value. Its minimum value is -2.

Q2 The graph is a parabola. If students are not familiar with the correct term, they should still be able to describe its form as being roughly U-shaped.

GRAPH A FUNCTION

Q3 Changing parameter a stretches the graph vertically, but the y-intercept does not change.

When $a = 0$, this eliminates the first term from the quadratic function, so it becomes a linear function, $f(x) = bx + c$.

When $a > 0$, the curve opens upward. When $a < 0$, it opens downward.

Q4 When you change b, the graph retains its shape, but it is translated along a curved path. The y-intercept does not change.

When $b = 0$, the curve is symmetric with respect to the y-axis.

Q5 Changing parameter c translates the graph vertically.

When $c = 0$, the curve passes through the origin.

Q6 When you change a or b, the y-intercept does not change. On the y-axis, $x = 0$. Substitute 0 for x in the equation $y = ax^2 + bx + c$, and the result is $y = c$. Therefore it does not matter what values a and b have. The y-intercept is at $(0, c)$.

EXPLORE MORE

11. These values are used here in order to ensure that the function has real roots.

12. Students may have trouble with some of the more complicated calculations. Advise them not to delete anything if they get it wrong. They can double-click a calculation to edit it, even after they have plotted the point.

Q7 The points are, in order, the vertex, the y-intercept, and the two x-intercepts.

In this activity students will observe the behavior of a plotted quadratic function in general form $f(a) = ax^2 + bx + c$ as you vary the three parameters a, b, and c. Students will form and evaluate conjectures about the effect of each of the parameters on the graph.

Use the prepared presentation sketch to eliminate some time-consuming parts of the student activity such as the entry of coordinate calculations.

1. Open **Graphing Quadratic Present.gsp.** Drag the sliders to show students how they control parameters a, b, and c.

The graph should appear after step 2. If it does not, the parameters may have moved it out of view.

2. Choose **Graph | Plot New Function.** The Calculator that appears has a key for the variable x. Enter the expression below. Enter the parameters by clicking on them.

$$ax^2 + bx + c$$

Special properties appear when any parameter is zero. There are action buttons to make it easier to hit that value.

Q1 Tell students that you are going to change the value of parameter a gradually. Ask them what effect that will have on the graph. After some discussion, drag the a slider. Give special attention to the linear graph that appears when $a = 0$. Guide them to explain this by substituting zero for a in the function.

Q2 Challenge students to predict the effects of changing b. This is more difficult to predict or explain. Show the y-axis symmetry that appears when $b = 0$.

Q3 Show and discuss the changes that result from dragging the c slider. Again, stop briefly at zero so students can see that the curve goes through the origin.

Q4 There is one point that is always on the curve when parameters a or b are changed. If students did not notice, go back through those motions, and tell them to watch for it. The invariant point is the y-intercept. Help them explain this by substituting zero for x in the function. The result is c, no matter what the values of the other parameters are.

Q5 Plot each of the points below. In each case, challenge students to predict the location of the plotted point. To save time with the calculations, press the *Show Coordinates* button. For each point, select the coordinates in order and choose **Graph | Plot As (x,y).**

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right); \quad (0, c); \quad \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right); \quad \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$$

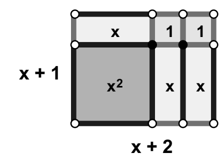
Factoring Trinomials

In this activity you'll factor trinomials visually by using Sketchpad algebra tiles.

SKETCH AND INVESTIGATE

1. Open **Factoring Trinomials.gsp.**

Q1 How many tiles are there for the trinomial $x^2 + 3x + 2$? How many different kinds? How do these tiles represent this particular trinomial?



2. To factor the trinomial, you must arrange its tiles into a rectangle. Press the *Arrange Tiles* button to do so.

Q2 What is the length of the resulting rectangle in terms of x? What is the width? Multiply these two binomials. What result do you get?

Q3 What are the factors of the trinomial $x^2 + 3x + 2$?

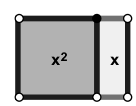
Use Sketchpad's Calculator to calculate the two results. With the Calculator open you can click the value of *x* in the sketch to enter it into your calculation.

Q4 Calculate the numeric result both ways to make sure the trinomial really is equal to the product of its factors. Record the value of x and the two results.

Q5 Page 2 contains four rectangles made from tiles. Write the area of each rectangle in trinomial and factored form. Drag the sliders to make sure the relationships hold for different values of the x, y, and *unit* variables.

On page 3 is another trinomial to factor. You will make this rectangle yourself.

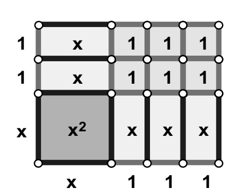
3. Drag the x^2 tile to an empty area of the sketch.



4. Move one of the x tiles so it's near the x^2 tile. Then drag the black point of the x tile to see how you can orient it either vertically or horizontally. Leave it vertical.
5. Drag the x tile so it's aligned on the right side of the x^2 tile.
6. Drag the remaining tiles so they are aligned with the tiles you have already placed and so all the tiles form a rectangle. You may need several tries to make this work.
7. When the tiles form a rectangle, use the **Text** tool to show the labels of each of the segments along the left and bottom edges of the rectangle.

Q6 What are the length and width of the rectangle? What are the factors?

To use the **x^2** custom tool, press and hold the **Custom** tools icon and choose **x^2** from the menu that appears.



8. Page 4 has another trinomial, but no tiles. Choose the **x^2** custom tool and click once in the sketch to create an x^2 tile.

Q7 How many x tiles will you need? Use the **x** custom tool to make them.

9. Use the **unit** tool to make as many unit tiles as you need, and then arrange the tiles into a rectangle.

Q8 What are the factors of this trinomial?

Even if you change the value of x, the tiles should still fit into a rectangle.

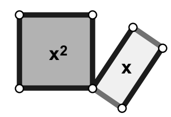
Q9 Make x smaller by dragging the blue slider to the left. What happens to the tiles?

Q10 Can you drag the tiles back into a rectangle? What are the factors now? Did changing the value of x change the factors?

On the next page you will attach the tiles so that they don't come apart.

10. On page 5 is another trinomial. Use the **x^2** tool to create the first tile.

11. Attach an x tile to the right side of the x^2 tile by clicking the **x** custom tool on the lower right vertex of the x^2 tile. Use the **Arrow** tool to drag the black point so that the x tile is oriented vertically.



12. Attach an x tile to the top of the x^2 tile by clicking the **x** tool on the upper left vertex of the x^2 tile. Use the **Arrow** tool to orient this new tile horizontally.

If you attach a tile in the wrong place, use **Edit** | **Undo** and then try again.

Q11 Construct the remaining tiles, attaching each new tile to the ones you have already placed. What are the factors of this trinomial?

Q12 Drag the x slider to change the value of x. What happens to the tiles?

Q13 Build and factor the following expressions on the remaining pages of the sketch. Draw the models on your paper.

a. $x^2 + 7x + 6$ b. $y^2 + 6y + 8$ c. $y^2 + 8y + 12$

d. $x^2 + 4xy + 4y^2$ e. $2x^2 + 3x + 1$ f. $4y^2 + 7y + 3$

Q14 On page 12, try to factor $x^2 + 4x + 6$. Describe the problem you encounter.

Q15 Describe how the second and third coefficients of a trinomial are related to the factors when the leading coefficient (the coefficient of the x^2 term) is 1.

EXPLORE MORE

Q16 Use the algebra tiles to factor the binomial $x^2 + 3x$.

Q17 Is there more than one rectangular shape that can be made to model any of the factorable expressions in this activity? See if you can come up with a factorable expression for which more than one rectangle can be made.

Objective: Students model the factoring of trinomials using Sketchpad's algebra tiles, explore why some trinomials can be factored and others cannot, and investigate the relationship between the factors and the coefficients of the trinomial.

Student Audience: Algebra 1

Prerequisites: Students should have some experience with multiplying binomials. The activity will be easier if students have already done an activity using algebra tiles (Algebra Tiles, The Product of Two Binomials, or Squaring Binomials).

Sketchpad Level: Intermediate. Students manipulate a prepared sketch and use custom tools.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Factoring Trinomials.gsp** in either setting)

The advantage of Sketchpad's algebra tiles is that they can be attached to each other and are based on variables rather than being of fixed size. This allows students to drag the sliders for x and y to see that x and y are variables and that the relationships work no matter what their values.

SKETCH AND INVESTIGATE

Q1 There are six tiles of three types, with each term's coefficient determining the number of tiles of that type.

Q2 The length is $(x + 2)$ and the width is $(x + 1)$. When you multiply these, you get the original trinomial $x^2 + 3x + 2$.

Q3 The factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$.

Q4 Answers will vary depending on the value of x.

Q5 $3x + 6 = 3(x + 2)$

$x^2 + 7x + 12 = (x + 4)(x + 3)$

$2y^2 + 5y + 2 = (y + 2)(2y + 1)$

$y^2 + 3xy + 2x^2 = (y + 2x)(y + x)$

Q6 The sides of the rectangle are $(x + 2)$ and $(x + 3)$. Some students will have $(x + 2)$ as the length and $(x + 3)$ as the width, and others will have it vice versa.

Q7 You'll need four x tiles.

Q8 The factors of $x^2 + 4x + 3$ are $(x + 3)$ and $(x + 1)$.

Q9 When you make x smaller, the tiles become smaller and spaces appear between them.

Q10 You can drag the tiles back into a rectangle. The factors remain the same: $(x + 3)$ and $(x + 1)$.

Q11 The factors of $x^2 + 5x + 4$ are $(x + 4)$ and $(x + 1)$.

Q12 When you drag the slider to change the value of x, the tiles get larger or smaller, but they remain attached in the form of a rectangle.

Q13 a. $(x + 6)(x + 1)$ b. $(y + 4)(y + 2)$

c. $(y + 6)(y + 2)$ d. $(x + 2y)(x + 2y)$

e. $(2x + 1)(x + 1)$ f. $(4y + 3)(y + 1)$

In all cases, the two factors can be reversed.

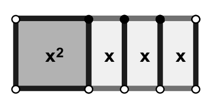
Q14 You cannot make a perfect rectangle with this group of tiles. The x's can all be lined up along one side of x^2, or three along one side and one along the other, or two along either side. These are the only possibilities, and in none of these cases do the six unit squares fit in.

Q15 The second term is the sum and the third term is the product of the same two numbers. If the factors are $(x + 4)$ and $(x + 2)$, the coefficient of the second term is the sum of 4 and 2, and the third term is the product of 4 and 2. Thus

$$x^2 + 6x + 8 = (x + 4)(x + 2)$$

EXPLORE MORE

Q16 You factor this binomial the same way, by making the tiles into a rectangular shape.



The factors are x and $(x + 3)$.

Q17 The only expressions that can be represented by more than one differently shaped rectangle have a constant factor common to each term. (This activity contains no such expressions.) An example is $2x^2 + 12x + 16$. This expression can be factored and represented with algebra tiles as either $(2x + 8)(x + 2)$ or $(2x + 4)(x + 4)$. Neither of these forms is fully factored. The full factorization is $2(x + 4)(x + 2)$. A model with algebra tiles would consist of two separate, identically sized rectangles.

In this presentation students will see how algebra tiles model factoring of trinomials, will see why some trinomials cannot be factored, and will see that the relationships explored hold even when the values of the variables are changed.

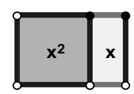
1. Open **Factoring Trinomials.gsp.** Ask students how the tiles correspond to the coefficients of the trinomial.

Q1 Press the *Arrange Tiles* button. After the rectangle appears, ask students to identify the length and width of the rectangle. Also ask them to identify the parts of the rectangle that correspond to the terms of the original trinomial.

Q2 Ask students to write the area in two different ways. Length times width gives $(x + 2)(x + 1)$, and counting the shapes gives $x^2 + 3x + 2$.

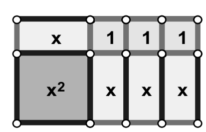
Choose **Measure | Calculate** to make the Calculator appear. To enter *x* into your calculation, click the value of *x* in the sketch.

2. Use the Calculator to make sure that both ways have the same numeric result.
3. Drag the x (blue) slider. The rectangle comes apart. Press the *Arrange Tiles* button to put it back together. On page 4 you'll make one that stays together.
4. On page 2, ask students to give the unfactored and factored expressions for the area of each rectangle.
5. On page 3, drag the x^2 tile to an empty area of the sketch.
6. Move one of the x tiles down so it's near the x^2 tile. Drag the black point of the x tile to show how you can orient it vertically or horizontally. Leave it vertical and drag the tile to align it on the right side of the x^2 tile.



Q3 Ask students how to drag each of the remaining tiles to form a rectangle. Have them summarize in their own words the factorization of this trinomial.

7. On page 4, choose the **x^2** custom tool and click once to create an x^2 tile.
8. Choose the **x** tool and click on the lower right corner of the square. Use the **Arrow** tool to make the x tile vertical.
9. Use the **x** tool and the **unit** tool to complete the rectangle.



Q4 Ask, "What is the factorization of $x^2 + 4x + 3$?"

Q5 Have students volunteer to use the tools to factor the problems on pages 5–10.

Q6 On page 11, have students write trinomial and factored form for each rectangle.

Q7 On page 12, try to factor $x^2 + 4x + 6$. What goes wrong?

Finish with a class discussion on factoring and on how the algebra tiles help illustrate the equivalence of the two forms.

Graphing Factored Quadratics

When you use quadratic functions to model real-world situations, you often begin with the roots. The roots are the x values for which the function is equal to zero. For the flight of a ball, the roots could represent the horizontal locations where it left the ground and where it landed. In cases like this, you use a quadratic function in factored form: $f(x) = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the two roots.

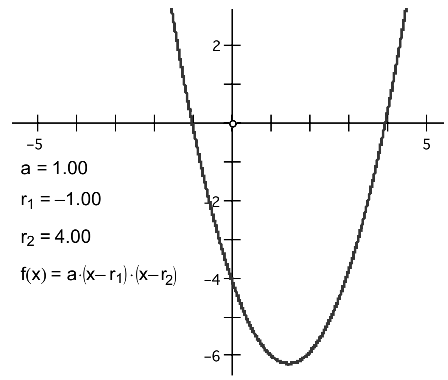
SKETCH AND INVESTIGATE

1. Open a new sketch. Choose **Graph | Define Coordinate System.**

The square brackets indicate a subscript. The labels *r[1]* and *r[2]* will appear as r_1 and r_2 in the sketch.

2. Choose **Graph | New Parameter** and label the new parameter *a.* Create a second parameter, labeling it *r[1]* and giving it a value of −1. Create a third parameter *r[2]* with a value of 4.

3. Choose **Graph | Plot New Function.** The Calculator dialog box that appears has a key labeled *x.* Enter $a(x - r_1)(x - r_2)$. To enter a, r_1, or r_2, click the parameter in the sketch.



Q1 What are the x-intercepts of the graph? Explain why this is true.

To make a parameter change more gradually, select it and choose **Edit | Properties | Parameter.** Change the Keyboard (+/−) Adjustments to 0.1.

4. Select one of the parameters. Change it by pressing the + and − keys. Experiment with all three parameters—a, r_1, and r_2.

Q2 Describe what happens when you change parameter a. How does the sign of a influence the shape of the graph? What happens when $a = 0$?

Q3 What happens when you change r_1 or r_2? What special property does the graph have when the two roots are equal?

Q4 You may know the roots of a certain quadratic function (as in the ball example), but that information alone is not enough to derive the function and draw the graph. Explain why.

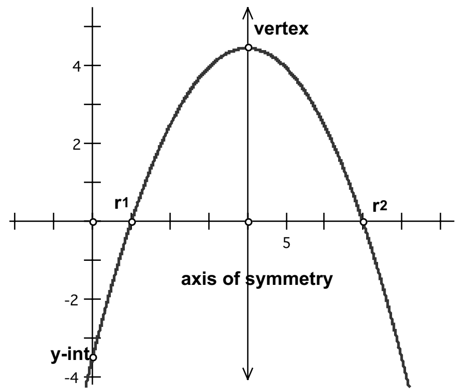
EXPLORE MORE

In this section you will use algebraic and geometric concepts together in order to plot other objects related to the graph.

The line of symmetry of this graph is vertical, and it must intersect the x-axis at a point midway between the two roots. Therefore, the x-coordinate of this point must be the mean of the two roots.

5. Calculate the mean by choosing **Measure | Calculator** and entering $(r_1 + r_2)/2$.

6. Create another new parameter. Label it *zero,* and set its value equal to zero.



7. Plot the point $((r_1 + r_2)/2, 0)$. To do so, select in order the calculation $(r_1 + r_2)/2$ and the parameter *zero,* and choose **Graph | Plot As (x,y).**

8. Construct the axis of symmetry by constructing a line parallel to the y-axis through the plotted point.

To construct the parallel, select the plotted point and the y-axis. Choose **Construct | Parallel Line.**

The vertex is on the axis of symmetry, so its x-coordinate must be $(r_1 + r_2)/2$. It is also on the parabola, so its y-coordinate is $f((r_1 + r_2)/2)$.

9. Calculate the y-coordinate of the vertex by choosing **Measure | Calculator.** Click the function $f(x)$ in the sketch and then the calculation $(r_1 + r_2)/2$. Click OK.

10. Plot the point $((r_1 + r_2)/2, f((r_1 + r_2)/2))$. Label it *vertex.*

Q5 What are the coordinates of the two x-intercepts and the y-intercept? Plot these points and label them appropriately.

11. Change the values of all three parameters, and observe the behavior of the intercepts, axis of symmetry, and vertex.

Q6 You can still change the parameters to show graphs of other quadratic functions. Are there any quadratic functions that you cannot show using this sketch? Explain.

Graphing Factored Quadratics

Objective: Students graph a function in the factored form $f(x) = a(x - r_1)(x - r_2)$ using the three parameters. They then investigate the relationships between the parameters and the graph.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should have already been introduced to quadratic functions. It's helpful if they're already familiar with the meaning of the term *root*.

Sketchpad Level: Intermediate. Students perform various calculation and graphing tasks. The Explore More section is considerably more difficult than the first part.

Activity Time: 30–40 minutes. The first section, Sketch and Investigate, can stand alone. This will reduce the time required by about half.

Setting: Paired/Individual Activity (no prepared sketch) or Whole-Class Presentation (use **Factored Quadratic Present.gsp**)

Related Activity: Graphing Quadratic Functions

SKETCH AND INVESTIGATE

Q1 The x-intercepts are -1 and 4, the same as the roots. If you substitute either of these numbers for x in the function definition, one of the factors will be zero, so the corresponding point must be on the x-axis.

Q2 Changing parameter a stretches the graph vertically from the x-axis while the x-intercepts remain constant. When $a > 0$, the parabola opens upward. When $a < 0$, it opens downward. When $a = 0$, the graph coincides with the x-axis. This is because the function definition becomes $f(x) = 0$.

Q3 When you change the roots, the x-intercepts change accordingly. When the roots are equal, there is only one intercept, and the parabola is tangent to the x-axis at that point.

Q4 If you know the roots, that gives you parameters r_1 and r_2, but you still need to know a in order to complete the function definition.

EXPLORE MORE

This last section guides students through some calculations and constructions. They must then finish with less guidance.

Q5 The coordinates of the x-intercepts are $(r_1, 0)$ and $(r_2, 0)$. The y-intercept is at $(0, f(0))$, or $(0, ar_1r_2)$.

Q6 In order to write a quadratic function in factored form, with real numbers, it must have roots. If the graph does not intersect the x-axis, this method will not work. One simple example is the function $f(x) = x^2 + 1$.

In this presentation students will observe the graph of a quadratic equation in factored form:

$$f(x) = a(x - r_1)(x - r_2)$$

Students will make conjectures about the behavior of the graph and will confirm or modify those conjectures as the parameters a, r_1, and r_2 change. They will see how to find the axis of symmetry, vertex, and intercepts from this form of the equation.

1. Open **Factored Quadratic Present.gsp.** The three sliders control the parameters a, r_1, and r_2. Demonstrate how the sliders control the parameters. Tell the class that during the construction, they should be watching for the relationships between these three parameters and the graph.

2. Choose **Graph | Plot New Function.** Enter $a(x - r_1)(x - r_2)$.

For the *x,* use the *x* key on the Calculator keypad. To enter a parameter, click it in the sketch.

Q1 Ask the class for the values of the x-intercepts. They are the roots. Drag the root sliders one at a time so that students can see that each slider changes only one root. To prove this, substitute one of the roots into the function definition, and show that it becomes zero.

Q2 Ask students to predict what will happen when you change parameter a. If they need a hint, ask them if changing a will change the x-intercepts. Drag the a slider slowly to show the family of parabolas having the given roots. Spend a moment on the special case of $a = 0$.

Q3 Ask what will happen if the roots are equal. Press the *One Root* button to move r_1 to r_2. The curve will be tangent to the x-axis at the one root.

Q4 Ask for the coordinates of the x-intercepts. They are $(r_1, 0)$ and $(r_2, 0)$.

3. Choose **Graph | New Parameter.** Change the label to *zero,* and make the value 0.

4. Select in order parameters r_1 and *zero.* Choose **Graph | Plot As (x,y).** Use the same method to plot $(r_2, 0)$.

5. Calculate and plot the y-intercept $(0, f(0))$ and vertex $((r_1 + r_2)/2, f((r_1 + r_2)/2))$. You can use the Calculator to compute $f(0)$ by clicking the function itself in the sketch and then entering the argument (0 in this example).

Q5 Compare this function with a quadratic function in general form $f(x) = ax^2 + bx + c$. What are the advantages of the different forms? Is it possible to convert one form to the other?