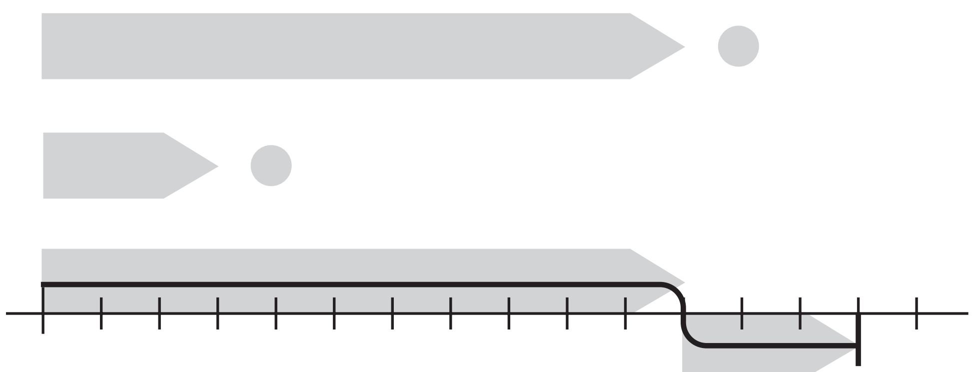
1

Fundamental Operations

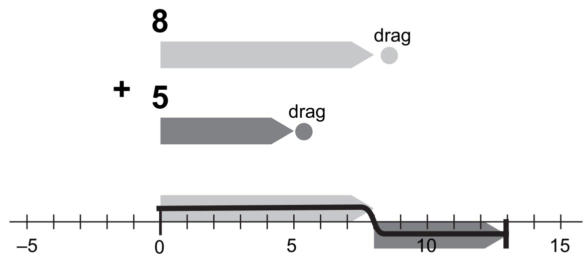


Adding Integers

In this activity you'll add integers using an animated Sketchpad model.

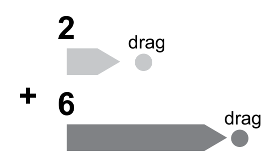
INVESTIGATE

1. Open **Adding Integers.gsp.** This sketch models the addition problem 8 + 5.
2. Press the *Present All* button to see the model in action.



Q1 How does the final position of the arrows show the answer for 8 + 5?

3. Press the *Reset* button, and then drag the circles to model 2 + 6.
4. This time, show the animation step by step: Press the *Show Steps* button, and then press each numbered button in order.



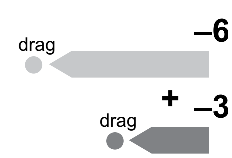
For each problem, press the buttons to show the result.

Q2 Drag the circles and press the buttons to model two other addition problems using only positive integers. Record each problem and the result.

Q3 How do the two upper arrows in the sketch relate to the two lower arrows?

Q4 Model $-6 + (-3)$. What's the sum?

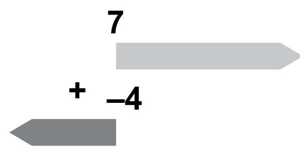
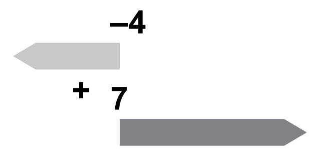
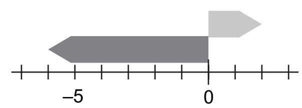
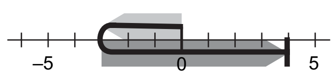
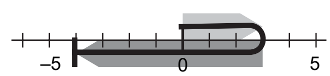
Q5 Model two more addition problems using negative integers. Record each problem and its result.



Q6 How is adding two negative numbers similar to adding two positive numbers? How is it different?

Q7 Can you add two negative numbers and get a positive sum? Explain.

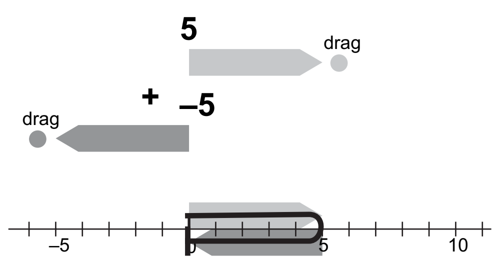
Q8 Model the following eight problems. Record each problem and its answer.

	
	
	
2 + (–5)	–2 + 5

Q9 When you add a positive and a negative integer, how can you look at the numbers and tell whether the answer will be positive or negative?

EXPLORE MORE

Q10 Model four problems for which the sum is zero. Make the first number positive in two problems and negative in two problems. Write down the problems you used. What must be true about two numbers if their sum is zero?



Q11 When you add two numbers, does the order matter? In other words, is $-3 + 5$ the same as $5 + (-3)$? Using the sketch, explain why your answer makes sense.

Objective: Students use an animated Sketchpad model for adding integers on the number line. They investigate addition of two positive numbers, addition of two negative numbers, and addition of a positive and a negative number.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This will be review for most Algebra 1 students.

Sketchpad Level: Easy. Students manipulate a prepared sketch.

Activity Time: 20–30 minutes. You may want to combine this activity and the Subtracting Integers activity in a single class period.

Setting: Paired/Individual Activity (use **Adding Integers.gsp**) or Whole-Class Presentation (use **Adding Integers Present.gsp**)

Use this activity as an introduction to integer addition for pre-algebra students, as a start-of-the-year refresher for Algebra 1 students, or as a supplemental activity for any student having difficulty with the topic. It's important for students to have a mental image of operations on integers. Even strong students who rely on verbal rules make careless mistakes that could be avoided by having an internalized picture.

The picture of addition presented here is a geometric model in which each number is represented by a vector. (The activity calls them *arrows* because students may not be familiar with the term *vector.*) Vectors incorporate both magnitude and direction (representing the absolute value and the sign of the integer), so practice with this model helps students understand how the signs of the addends come into play.

This activity contains lots of questions for students, who develop their understanding through the process of manipulating the sketch and describing what they observe. Encourage them to write clear and detailed explanations (and to use complete sentences) when they answer the questions; the extra time it takes them to do so is well spent.

If there's time and you have a presentation computer with a projector, have different students use Sketchpad to demonstrate to the class their observations or the problems they made up. It's a big help to students if they can listen to, evaluate, and discuss the descriptions and conclusions of their classmates.

INVESTIGATE

Students may be unfamiliar with *model* as a transitive verb; consider reviewing with them the various uses of this word.

Q1 In their final positions, the second arrow starts from where the first arrow ends, and the answer (13) is at the end of the second arrow. Encourage students to be detailed and specific in their answer to this question.

Q2 Answers will vary but should include only positive numbers.

Q3 Each lower arrow is exactly the same size and direction as the corresponding upper arrow.

Q4 The sum of $-6 + (-3)$ is -9.

Q5 Answers will vary but should include only negative numbers.

Q6 Whether adding two negative or two positive numbers, both arrows go the same way, taking the sum farther away from the center of the number line (farther away from zero). The difference is that the arrows go to the right when the numbers are positive but go to the left when they're negative.

Q7 When you add two negative numbers, you cannot get a positive sum. Both numbers take the sum in the negative direction from zero, so the sum must be negative.

Q8 As students model various problems, walk around the room and observe them to make sure they can model any problem they are given.

$7 + (-4) = 3$	$-4 + 7 = 3$
$-6 + 2 = -4$	$2 + (-6) = -4$
$-3 + 7 = 4$	$3 + (-7) = -4$
$2 + (-5) = -3$	$-2 + 5 = 3$

Q9 When you add a positive and a negative integer, the number that has the larger absolute value tells you whether the answer will be positive or negative. In other words, the sign of the result is the same as the sign of the longer arrow.

EXPLORE MORE

Q10 Each student will model different problems. In every case, the two numbers must be opposites, so that their arrows are the same length but point in opposite directions.

Q11 The order does not matter when you add two numbers. The arrows determine how far you go and in which direction, and it doesn't matter if you follow the first arrow and then the second, or if you follow the second arrow and then the first.

WHOLE-CLASS PRESENTATION

Start the whole-class presentation by animating the addition of two positive integers (Q1–Q3 of the activity). Open the sketch **Adding Integers Present.gsp** and press the step-by-step buttons one at a time, pausing between animations. Ask students to describe what they see as the animation progresses, and be sure to get observations from several different students. Press the *Reset* button, change the problem by dragging both circles (while leaving the numbers positive), and press the step-by-step buttons again.

Next animate the addition of two negative numbers (Q4–Q7 of the activity). Press *Reset,* drag the numbers so they are both negative, and ask students to predict what will happen now. Use the step-by-step buttons to test their conjectures. Without resetting, ask questions Q6 and Q7, and experiment by dragging to change the values of the numbers.

When students are satisfied with the results of adding two negative numbers, animate the addition of numbers with different signs. Reset again and drag the circles so one of the numbers is positive and one is negative. Ask students to predict how the arrows will behave. (Try to get students to concentrate on the behavior of the model rather than on the numeric answer.) Use the buttons again to show the behavior. Model several more problems (such as those in Q8) involving a positive and a negative number.

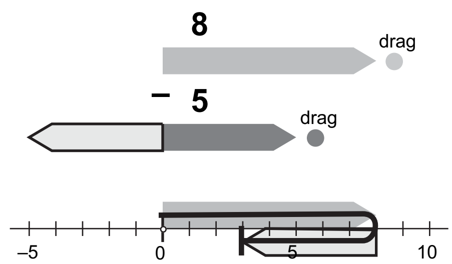
Finish the class discussion using Q9, Q10, and Q11. When students propose an answer to one of these questions, have them manipulate the sketch to show why their answer makes sense.

Subtracting Integers

In this activity you'll subtract integers using an animated Sketchpad model.

INVESTIGATE

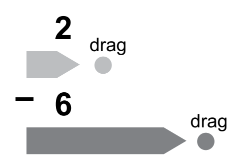
1. Open **Subtracting Integers.gsp.** The sketch models the subtraction problem $8 - 5$.
2. Press the *Present All* button to see the model in action.



Q1 During the animation, what happens to the arrow for 5?

Q2 How does the final position of the bottom arrows show the answer for this subtraction problem?

3. Press the *Reset* button, and then drag the circles to model $2 - 6$.



4. This time, show the animation step by step: Press the *Show Steps* button, and then press each numbered button in order.

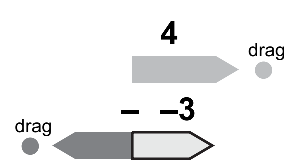
Q3 Describe in your own words what the *3. Make Inverse* step does.

For each problem, press the buttons to show the result.

Q4 Drag the circles to model two more subtraction problems that use positive integers but have a negative result. Record each problem and its result.

Q5 If both numbers in a subtraction problem are positive, how can you tell if the answer will be positive or negative?

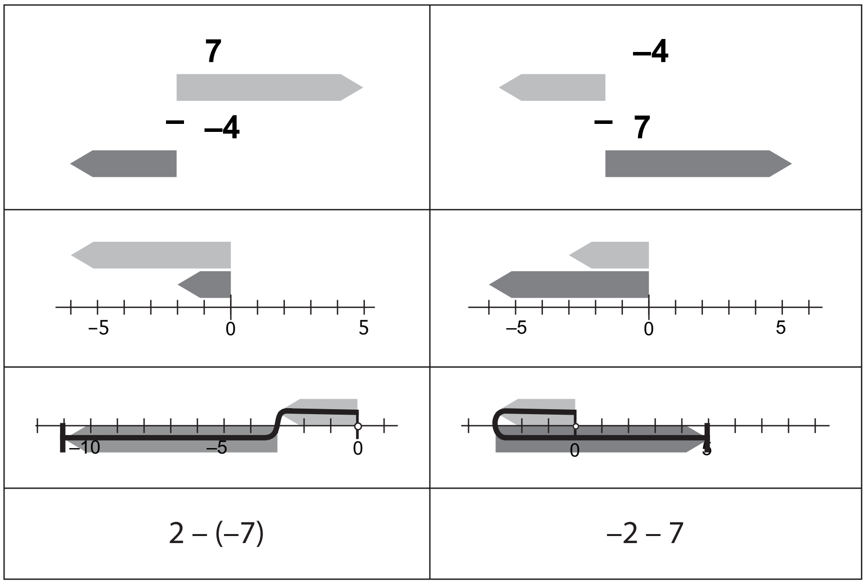
Q6 Model $4 - (-3)$. What's different about the *3. Make Inverse* step this time?



Q7 Model two more problems in which the first number is positive and the second number is negative. Record each problem. What do these models have in common?

Q8 Model three problems in which the first number is negative and the second number is positive. Record each problem. What do these models have in common?

Q9 Model the following eight problems. Record each problem and its answer.

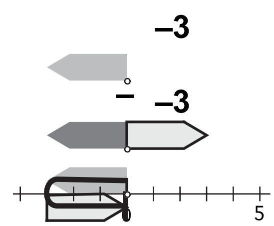


For instance, $7 - (-4) = 11$, so fill in the blank: $7 + \_\_\_ = 11$.

Q10 For each subtraction problem above, write an addition problem that has the same first number and the same answer. What do you notice?

EXPLORE MORE

Q11 Model four subtraction problems for which the difference is zero. Make the first number positive in two problems and negative in two problems. Write down the problems you used. What must be true about two numbers if their difference is zero?



Q12 Model four subtraction problems in which the difference is the same as the first number. What must be true of these problems?

Q13 Model four subtraction problems in which the difference is the same as the second number. What must be true of these problems?

Q14 When you subtract two numbers, does the order matter? In other words, is $-3 - (-5)$ the same as $-5 - (-3)$? Explain in terms of the model why your answer makes sense.

Objective: Students use an animated Sketchpad model for subtracting integers on the number line, and see the second number being flipped before it's added to the first number. Students investigate subtraction of two positive numbers and various subtraction problems involving negative numbers.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This will be review for most Algebra 1 students.

Sketchpad Level: Easy. Students manipulate a prepared sketch.

Activity Time: 20–30 minutes. You may want to combine this activity and the Adding Integers activity in a single class period.

Setting: Paired/Individual Activity (use **Subtracting Integers.gsp**) or Whole-Class Presentation (use **Subtracting Integers Present.gsp**)

Use this activity as an introduction to integer subtraction for pre-algebra students, as a start-of-the-year refresher for Algebra 1 students, or as a supplemental activity for any student having difficulty with the topic. It's important for students to have a mental image of operations on integers. Even strong students who rely on verbal rules make careless mistakes that could be avoided by having an internalized picture.

The picture of subtraction presented here is a geometric model in which each number is represented by a vector. (The activity calls them *arrows* because students may not be familiar with the term *vector.*) Vectors incorporate both magnitude and direction (representing the absolute value and the sign of the integer), so practice with this model helps students understand how the signs of the operands come into play.

The questions are critical in encouraging students to internalize the model presented in this activity. Make sure students write clear and detailed explanations (and use complete sentences) when they answer the questions; the extra time it takes them to do so is time well spent.

If there's time and you have a presentation computer with a projector, have different students use Sketchpad to demonstrate to the class their observations or the problems they made up. It's a big help to students if they can listen to, evaluate, and discuss the descriptions and conclusions of their classmates.

INVESTIGATE

These notes sometimes use the terms *minuend* (first number) and *subtrahend* (second number), but these terms are not used in the student material. If you do use them with students, be sure to explain them carefully.

The concept of *additive inverse* is not named, but it plays a prominent role in the animation. You should discuss with the class why the second number must be flipped, even if you don't give a name to that operation.

Q1 During the animation, the arrow for 5 flips from the right to the left. This shows which way the second arrow must go in order to subtract it from the first.

Q2 In their final positions, the flipped second arrow starts from where the first arrow ends, and the answer (3) is at the end of the second arrow. Encourage students to be detailed and specific in their answer to this question.

Q3 Answers will vary. Students should describe the arrow flipping from right to left; encourage them to explain in their own words why it needs to flip in order to do subtraction.

Q4 Answers will vary but should include only problems in which a positive minuend is smaller than a positive subtrahend.

Q5 If both numbers are positive, the result will be positive if the first number (minuend) is larger, and negative if the second number (subtrahend) is larger.

Q6 Some students will record direct observations, and others will interpret those observations. Typical answers will be similar to the following.

Observation: In this problem, $4 - (-3)$, the second arrow starts out pointing to the left, so when it flips it turns around and points to the right.

Interpretation: The second number starts out negative, so when it flips it becomes positive.

Q7 The problems students create will vary. Because the first number is positive and the second negative, the models have in common that, after flipping, both arrows point to the right, and the result must be positive.

Q8 Problems will vary. Because the first number is negative and the second positive, after flipping, both arrows point to the left, and the result is negative.

Q9 As students model various problems, walk around the room and observe them to make sure they can model any problem they are given.

$7 - (-4) = 11$ $\quad -4 - 7 = -11$

$-6 - (-2) = -4$ $\quad -3 - (-6) = 3$

$-3 - 8 = -11$ $\quad -3 - (-8) = 5$

$2 - (-7) = 9$ $\quad -2 - 7 = -9$

Q10 Written as addition problems, these problems become

$7 + 4 = 11$ $\quad -4 + (-7) = -11$

$-6 + 2 = -4$ $\quad -3 + 6 = 3$

$-3 + (-8) = -11$ $\quad -3 + 8 = 5$

$2 + 7 = 9$ $\quad -2 + (-7) = -9$

In each case, to subtract you can change the sign of the second number and add them. This is similar to the way the second arrow flips before the animation shows the answer.

EXPLORE MORE

Q11 For a subtraction problem to have an answer of zero, the two numbers being subtracted must be the same.

Q12 To make the difference the same as the first number, the second number must be zero.

Q13 To make the difference the same as the second number, the first number must be twice as big as the second. For instance, $6 - 3 = 3$, and $-8 - (-4) = -4$.

Q14 The order does matter when you subtract numbers, because only the second arrow is flipped. More sophisticated students will observe that the order matters only if the second number is nonzero, because flipping zero has no effect.

WHOLE-CLASS PRESENTATION

Start the whole-class presentation by animating the subtraction of two positive integers (Q1–Q5 of the activity). Open the sketch **Subtracting Integers Present.gsp** and press the step-by-step buttons one at a time, pausing between animations. Ask students to describe what they see as the animation progresses, and be sure to get observations from several different students. Press the *Reset* button, change the problem by dragging both circles (while leaving the numbers positive), and press the step-by-step buttons again. Pay special attention to Q3 and Q5.

Next animate subtraction problems in which the first number is positive and the second number is negative (Q6–Q7 of the activity). Press *Reset,* make the first number positive and the second negative, and ask students to predict what will happen now. Test their conjectures using the step-by-step buttons. Repeat for several more problems.

Animate subtraction problems like those in Q8 and Q9, and record the answers for each of the problems in Q9. Ask students what patterns they see, and how they could predict the answer from the two numbers being subtracted.

For Q10, ask students to make an addition problem for each of the problems from Q9, and test their addition problems using page 2 of the sketch. Switching back and forth between page 1 and page 2 will reinforce for students the idea of using addition to rewrite a subtraction problem.

Continue the class discussion with as many of the Explore More questions (Q11–Q14) as are appropriate for the class and the available time.

Finish by having students summarize in their own words the relationship between subtraction and addition.

Raz's Magic Multiplying Machine

"Step right up, folks! Have I got a machine for you. You've seen number lines, right? I don't mind telling you, all those other number lines lack pizzazz . . . or, should I say, Raz-matazz?

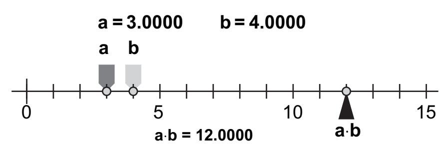
"Say hello to my latest state-of-the-art number line. For starters, give it any two numbers and it multiplies them. Nifty, eh?

"But there's more. The eye-popping things this number line does will change the way you think about multiplication. Come give it a whirl!"

STATIC MULTIPLICATION

You can use the right and left arrow keys on your keyboard to move a selected marker one pixel at a time.

1. Open **Multiplying Machine.gsp.** Drag markers a and b to see how they behave and how they affect marker $a \cdot b$.



2. Drag markers a and b to represent the problem $3 \cdot 4 = 12$.

The product marker, $a \cdot b$, is to the right of the factors, a and b. This makes sense, since the product of two numbers greater than 1 is always bigger than either factor.

You can change the scale of the number line by showing the number line controls and either pressing the *Unit Distance* buttons or dragging point *scale.*

Q1 Drag a to 0.5 and b to 6. What does the machine show for the product $0.5 \cdot 6$?

Q2 List four pairs of locations for a and b such that $a \cdot b = 6$.

Q3 Find a location for a and b in which all three markers lie directly on top of each other. Is there more than one location that works?

Q4 If b and $a \cdot b$ are the same distance away from 0, but on opposite sides, where must a be?

Q5 Can a, b, and $a \cdot b$ all lie to the left of 0? Explain.

Q6 Find locations for a and b to the right of 0 such that $a \cdot b$ is smaller than both a and b. Describe all locations for a and b for which this works.

DYNAMIC MULTIPLICATION

When you multiply two numbers on a calculator, the only thing you see is the answer. Raz's machine gives the answer, but as you drag a or b toward its intended value, you get to observe the product, $a \cdot b$, as it moves simultaneously. This is fun to watch, and it can deepen your understanding of multiplication.

3. Drag a to 0. Then slowly drag b back and forth along the number line.

Q7 What happens to the product? Why does this make sense?

Q8 Drag a to 1. Then drag b back and forth along the number line. Describe the movement of $a \cdot b$ in relation to b. Why does this behavior make sense?

Q9 Drag a to -1. Then drag b back and forth along the number line. Describe the movement of $a \cdot b$ in relation to b. Why does this behavior make sense?

Q10 Drag a to 0.5. Then drag b back and forth along the number line. Which moves faster, b or $a \cdot b$? Explain why.

Q11 Find a location for point a such that the distance from $a \cdot b$ to 0 is always twice the distance from b to 0. Are there other answers?

WHY IS A NEGATIVE TIMES A NEGATIVE A POSITIVE?

Have you ever wondered why a negative number times a negative number is a positive number? Raz's machine provides a nice way to visualize the reason.

Remember you can use the arrow keys to drag a slowly.

4. Move both a and b so that they're near the right edge of the sketch window. Now drag a slowly to the left, and watch $a \cdot b$ glide across the screen.

   When $a \cdot b$ reaches the left edge of the sketch, move a in the opposite direction so that $a \cdot b$ glides back to the right. Drag a back and forth while observing $a \cdot b$.

This type of explanation is sometimes called "reasoning by continuity."

Q12 Based on what you've observed, explain why it makes sense that a positive number times a negative number equals a negative number.

Q13 Move b to the left of 0. Once again, drag a back and forth, observing the behavior of the product, $a \cdot b$. Based on the behavior you observe, explain why it makes sense that a negative number times a negative number equals a positive number.

EXPLORE MORE

Go to the Construction pages and explore the first two designs Raz manufactured, using geometric technology. Try to duplicate his constructions. Then try to design and construct your own machine.

Objective: Students explore multiplication dynamically, dragging the input values and observing the behavior of the output value. As they observe the dynamic behavior of the values, students discover the special roles that zero and one play in multiplication and the rules regarding the sign of a product.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This is a review topic for most Algebra 1 students.

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Multiplying Machine.gsp**) or Whole-Class Presentation (use **Multiplying Machine Present.gsp**)

This is a useful activity, both for reviewing multiplication of positive and negative numbers and for developing number sense in general. Students can use this simple machine to explore many important number properties, such as multiplication of negatives, square roots, ratios and proportions, and multiplication by numbers between 0 and 1 (and also between -1 and 0). The key is for students to explain mathematically the behavior they observe: "What is it about multiplication that makes the machine behave like this?" Encourage students to answer the questions with detailed descriptions that include both observations and explanations, and encourage them to do their own explorations and observe their own patterns.

STATIC MULTIPLICATION

Q1 $0.5 \cdot 6 = 3$.

Q2 Answers will vary. In a class discussion, you might ask students what different *types* of answers to this question are possible. (One possibility is that both factors could be positive integers. Another is that one could be a positive integer and the other could be a decimal between 0 and 1.)

Q3 The only two possible locations are at 0 ($0 \cdot 0 = 0$) and 1 ($1 \cdot 1 = 1$). (To see that these are the only two solutions, solve the equation $x \cdot x = x$.)

Q4 a must be at -1.

Q5 No, all three markers cannot be to the left of 0. When a and b are to the left of 0, $a \cdot b$ is to its right, since a negative times a negative equals a positive. When $a \cdot b$ is to the left of 0, one of the factors is also to the left of 0 and the other is to the right, since a negative times a positive equals a negative.

Q6 Many possible answers. In all cases, both a and b are between 0 and 1. When students explore this question, they may find it beneficial to show the number line controls and change the unit distance to 50 pixels.

DYNAMIC MULTIPLICATION

Q7 Both a and $a \cdot b$ remain fixed at 0. This makes sense because any number times 0 equals 0. (This is the multiplication property of zero.)

Q8 b and $a \cdot b$ stay right on top of each other. This makes sense because any number times 1 equals itself. (This is the multiplication property of one: the number 1 is the *multiplicative identity.*)

Q9 b and $a \cdot b$ move in opposite directions and at the same speed. They can be thought of as reflections of each other across 0. This makes sense because any number times -1 equals its opposite—the number just as "big" but on the other side of 0.

Q10 b moves faster than $a \cdot b$—twice as fast, to be exact. This is because a number multiplied by a number between 0 and 1 gives a result smaller (closer to 0) than the original number.

Q11 The two possible answers are 2 and -2.

WHY IS A NEGATIVE TIMES A NEGATIVE A POSITIVE?

Q12 When b is positive, dragging a to the left also moves $a \cdot b$ to the left. As a approaches 0, $a \cdot b$ approaches 0 too, and when $a = 0$, $a \cdot b = 0$. It makes sense that as you keep dragging a to the left, $a \cdot b$ continues its previous behavior, moving to the left into negative territory.

Q13 With b to the left of 0, a and $a \cdot b$ move in opposite directions. As you drag a to the left, $a \cdot b$ moves to the right. As a approaches 0, $a \cdot b$ approaches 0 too, but from the other side, and when $a = 0$, $a \cdot b = 0$. It makes sense that as you keep dragging *a* to the left, $a \cdot b$ continues its previous behavior, moving to the right into positive territory.

EXPLORE MORE

Encourage students to explore one or both of the geometric constructions. Students will find Object Properties (and the Parent/Child pop-up menus) useful in understanding the existing constructions.

WHOLE-CLASS PRESENTATION

This whole-class presentation works well if you have a student operate the computer while you provide direction, ask questions, and lead the discussion.

Start the presentation by opening the sketch **Multiplying Machine Present.gsp** and asking students how the markers show the original multiplication problem. Ask them to predict what will happen if you drag marker *a* to the right, and what will happen if you drag it to the left. Drag the marker so they can test their predictions.

Continue by posing questions Q1–Q6 from the student activity sheet. (The presentation sketch has pages corresponding to each of the questions, although you may not need to use them.)

Pose questions Q7–Q11, paying particular attention to the machine's behavior when *a* has been dragged to values of 0, 1, and −1.

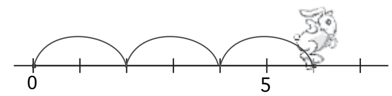
Finish the class discussion by analyzing what happens when you start with positive values for both *a* and *b,* and then slowly drag each in turn to the negative side of the number line. Ask students to explain in their own words how the machine's behavior makes sense when multiplying a positive times a negative (Q12), and why it makes sense when multiplying a negative times a negative (Q13). Elicit responses from a number of students, dragging the markers to illustrate their answers, so that various students get a chance to explain the machine's behavior in terms that make sense to them.

Multiple Models of Multiplication

What does *multiplication* mean? This question has many answers, because there are many ways of thinking about multiplication. In this activity you'll compare four such ways—multiplication as jumping, as grouping, as area, and as scaling.

MULTIPLICATION AS JUMPING

You can think of multiplication as jumping: Three jumps of two units each could be described by the multiplication problem 3 • 2. In this model, you will experiment with setting the number of jumps and the size of each jump.



1. Open **Multiplication Models.gsp.** Press the *Jump!* button to animate three jumps of two units each.
2. Press the *Reset* button, drag the circles to represent two jumps of five units each, and press the *Jump!* button again.

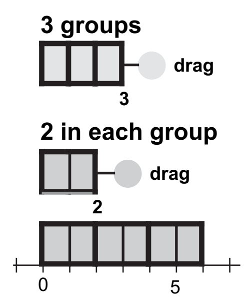
Q1 How many ways can you do jumps that end up at 6? Drag the green circles to try each way, and write down all the ways you found.

Q2 Change the number of jumps so that it's negative. What happens during the jumping? Make the number of jumps positive again, and make the size of each jump negative. What happens?

Q3 What happens if the number of jumps and the size of each jump are both negative? How can you explain this logically?

MULTIPLICATION AS GROUPING

You can also think of multiplication as grouping: 3 • 2 means three groups of two things each. In this model, you will group rectangles along a number line.



3. Go to the Grouping page. The objects in the sketch model the sentence "Put together three groups of two." The equation is $3 \cdot 2 = 6$.

Q4 Drag the circles to model each sentence below. On your paper, draw the bottom shape (the one on the number line) and write its equation.

a. Put together four groups of 2.

b. Put together three groups of -3.

c. Put together one group of -8.

d. Put together eight groups of -1.

How should you drag the top circle to represent "take away"?.

e. Take away two groups of 3.

f. Take away one group of 5.

g. Take away two groups of -3.

h. Take away eight groups of -1.

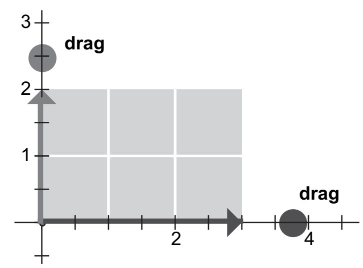
Q5 Model the following sentences and write their equations. How are they similar and how are they different?

a. Put together three groups of -4.

b. Put together four groups of -3.

Q6 Using 4's and 3's, write and model two "take away" sentences whose product is the same as the product in Q5.

MULTIPLICATION AS AREA



Another way to think about multiplication is in connection with the area of rectangles.

4. Go to the Area page. In this model, the height and width can be either positive or negative. When you start, both are positive.

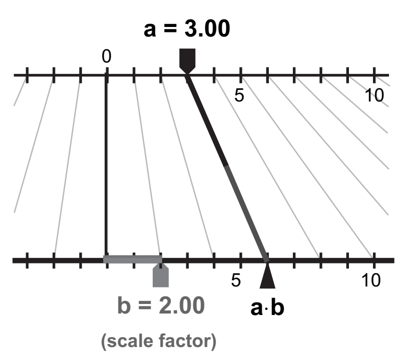
Q7 Drag the width to model $-3 \cdot 2 = -6$. What happens to the rectangle when the width becomes negative? What does this change indicate about the area?

Q8 Model seven different problems in which the area equals –6. Write the problems on your paper.

Q9 Model and write down as many problems as you can in which the area equals 4.

Q10 The numbers 1, 4, 9, 16, . . . are called "squares." Explain why this makes sense given the area model of multiplication.

MULTIPLICATION AS SCALING



Whether you're drawing a scale model of your room or scaling a recipe to serve more people, you're using multiplication.

5. Go to the Scaling page. The *scale factor* (b) is 2, so every number on the top axis *maps* (corresponds) to a number twice as big on the bottom axis. Point a is at 3 and maps to 6, so the equation for this problem is $3 \cdot 2 = 6$.

6. Drag a to model $-3 \cdot 2 = -6$. Then drag b to model $-3 \cdot (-2) = 6$.

Q11 Describe what the gray mapping segments look like when:

a. b equals 1.

b. b is between 0 and 1.

c. b equals zero.

d. b is negative.

Q12 For each problem below, set the scale factor b as listed, and then drag a so that $a \cdot b = 1$. (For example, if b were 0.5, you would make $a = 2$ because $0.5 \cdot 2 = 1$.)

a. $b = 4$; $a \cdot b = 1$; $a = ?$

b. $b = -0.5$; $a \cdot b = 1$; $a = ?$

c. $b = -1$; $a \cdot b = 1$; $a = ?$

d. $b = -10$; $a \cdot b = 1$; $a = ?$

Q13 Rewrite the answers to Q12 using fractions instead of decimals. What do you notice?

SUMMING UP

Q14 List one strength of each of the four models, perhaps something that each shows about multiplication better than the others.

Q15 Which of the four models do you think is most effective at showing why the product of two negatives is a positive? Defend your choice.

EXPLORE MORE

To copy from Sketchpad into a word processor, select the objects you want to copy, and resize the window to the desired size of your picture. Then choose **Edit | Copy,** and paste the result into your word processor.

Q16 The commutative property of multiplication says that it doesn't matter whether you multiply $3 \cdot 2$ or $2 \cdot 3$; you get the same answer using either order. Set up four pairs of multiplication problems, one for each model, to show this property. Copy the Sketchpad image for each of the eight problems and paste them into a word processor document. Which model do you think is most effective at showing why multiplication is commutative?

Objective: Students work with four different models of multiplication and use each model to solve problems and investigate properties of multiplication. Students compare the four models, particularly with regard to how they make sense of negative operands.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This will be a review topic for most Algebra 1 students, though perhaps it presents things in a new way.

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Multiplication Models.gsp** in either setting)

This activity has two main purposes: to provide students with multiple models of multiplication and to give a variety of justifications for the rules for multiplying negatives.

By using multiple models of multiplication, students consider different ways of conceiving of this key operation and gain perspective on its meaning and uses. For this reason, don't allow students to do the problems in their heads without modeling them in Sketchpad—that would defeat the purpose.

As they work, students see how each model provides justification for the rules for multiplying negatives. A mental image provides a more solid foundation than a verbal rule for the idea that the product of two negative numbers is positive.

Keep in mind that these four models of multiplication aren't the only ones in this chapter. The Raz's Magic Multiplying Machine activity provides yet another model that challenges students to broaden their thinking about multiplication and gives compelling reasons for the rules for multiplying negatives. These two activities work especially well together.

MULTIPLICATION AS JUMPING

Q1 Jumps that end up at 6 include $3 \cdot 2$, $2 \cdot 3$, $6 \cdot 1$, $1 \cdot 6$, $-3 \cdot (-2)$, $-2 \cdot (-3)$, $-6 \cdot (-1)$, and $-1 \cdot (-6)$.

Q2 When the number of jumps is negative and each jump is positive, the rabbit faces right and jumps backward, moving to the left. When the number of jumps is positive but each jump is negative, the rabbit faces left and jumps forward, again moving to the left.

Q3 When both the number of jumps and the size of each jump are negative, the rabbit faces left and jumps backward, moving to the right. He faces left because the size of the jumps is negative, and he jumps backward because he's taking a negative number of jumps. By facing left and jumping backward, the rabbit moves in the positive direction along the number line.

MULTIPLICATION AS GROUPING

Q4 a. $4 \cdot 2 = 8$ b. $3 \cdot (-3) = -9$

c. $1 \cdot (-8) = -8$ d. $8 \cdot (-1) = -8$

e. $-2 \cdot 3 = -6$ f. $-1 \cdot 5 = -5$

g. $-2 \cdot (-3) = 6$ h. $-8 \cdot (-1) = 8$

Q5 a. $3 \cdot (-4) = -12$ b. $4 \cdot (-3) = -12$

These two results are similar in that they both give the same negative answer, -12. In both cases, one number is positive and one is negative. The biggest difference is that the reason for changing direction, from positive to negative, is completely different in the two cases.

Q6 "Take away three groups of 4" ($-3 \cdot 4 = -12$) and "take away four groups of 3" ($-4 \cdot 3 = -12$).

MULTIPLICATION AS AREA

Q7 When the width becomes negative, the rectangle flips over horizontally, the squares change color, and the area becomes negative. Some students may make a valuable logical connection between the flipping of the rectangle and the area becoming negative.

Q8 $-1 \cdot 6 = -6$ $-2 \cdot 3 = -6$

$-6 \cdot 1 = -6$ $1 \cdot (-6) = -6$

$2 \cdot (-3) = -6$ $3 \cdot (-2) = -6$

$6 \cdot (-1) = -6$

Q9 $1 \cdot 4 = 4$ $2 \cdot 2 = 4$

$4 \cdot 1 = 4$ $-1 \cdot (-4) = 4$

$-2 \cdot (-2) = 4$ $-4 \cdot (-1) = 4$

Q10 Every square number can be modeled with a square in the area multiplication model. For example, 4 can be modeled by $2 \cdot 2$ or $-2 \cdot (-2)$, both of which are squares.

(A number such as -4 can also be modeled with squares, $2 \cdot (-2)$ or $-2 \cdot 2$. However, in these squares, the base and height are not equal. This can be interpreted as a weakness of this model, or it might represent an opportunity for a sneak preview of imaginary numbers.)

MULTIPLICATION AS SCALING

Q11 a. The mapping segments point straight down, parallel to each other. Every number maps to itself. For example, $2 \cdot 1 = 2$, $-3 \cdot 1 = -3$, $0 \cdot 1 = 0$, etc.

b. The mapping segments point inward toward the bottom. Every number maps to a number whose absolute value is less than its own absolute value (or equal to, in the case of 0), but whose sign is the same. For a scale factor of 0.5, for example, $2 \cdot 0.5 = 1$, $-3 \cdot 0.5 = -1.5$, $0 \cdot 0.5 = 0$, etc.

c. The mapping segments all point to zero, so every number maps to zero. For example, $2 \cdot 0 = 0$, $-3 \cdot 0 = 0$, $0 \cdot 0 = 0$, etc.

d. The mapping segments cross between the two number lines. Every number maps to a number with the opposite sign (except for 0, which points to itself). For a scale factor of -2, for example, $2 \cdot (-2) = -4$, $-3 \cdot (-2) = 6$, $0 \cdot 0 = 0$, etc.

Q12 a. $a = 0.25$ b. $a = -2$

c. $a = -1$ d. $a = -0.1$

Q13 a. $b = 4$; $a = 1/4$ b. $b = -1/2$; $a = -2/1$

c. $b = -1/1$; $a = -1/1$ d. $b = -10$; $a = -1/10$

In each pair, the numbers are reciprocals of each other. For example, in part a, $b = 4/1$ and $a = 1/4$.

SUMMING UP

Q14 There are many possible answers. We feel that Jumping and Grouping are particularly effective as an introduction to multiplication. They correspond with most people's basic conception of multiplication and so are a good place to start. Area is particularly effective at showing the "dimensionality" of multiplication—how multiplying two one-dimensional objects produces a two-dimensional object. Scaling is good for showing how multiplication affects an entire set of objects, including non-integers. It also serves as a great introduction to dynagraphs.

Q15 Jumping, Grouping (especially when using the terms "put together" and "take away"), and Scaling are effective at demonstrating the rules of multiplication for negatives. Area is less effective for this, in our view, because there is no compelling reason why the rectangles in the first and third quadrants are blue and those in the second and fourth quadrants are red.

EXPLORE MORE

Q16 Students should model pairs of equations, such as $2 \cdot (-5) = -10$ and $-5 \cdot 2 = -10$. Area may be especially useful for demonstrating commutativity because it's so easy to see that the two rectangles have the same area and sign.

WHOLE-CLASS PRESENTATION

The whole-class presentation of this activity substantially follows the steps of the student activity sheet. Refer to the Presenter Notes for tips to follow and adjustments to make so that the presentation can be as useful to students as possible.

You can present any of the models in this activity independently, though it's valuable to present at least two models in succession. Students will get the greatest benefit from this activity when they compare the behaviors of several different models.

Follows the steps in the student activity sheet, with the adjustments described below.

MULTIPLICATION AS JUMPING

Be sure to elicit answers from a number of students.

When the rabbit first jumps, ask students how the rabbit's motion illustrates the multiplication problem shown before adjusting the numbers (leaving both positive) and doing another example.

Before changing the number of jumps to be negative, ask students to predict what the rabbit will do.

Be sure to make the *jumps* value positive again before you make the *units* value negative.

Similarly, before making the size of each jump negative, ask students to predict what the rabbit will do. And ask again for predictions before making both numbers negative at the same time.

MULTIPLICATION AS GROUPING

In the grouping part of the presentation, ask students to make up problems using particular combinations of negative and positive ("put together" and "take away") rather than using the specific ones from Q4. Be sure to show how a "put together" problem and a "take away" problem can give the same result.

MULTIPLICATION AS AREA

When presenting multiplication as area, you may want to emphasize that the color of the rectangle indicates the sign of the result, without too much emphasis on the idea of "negative area." Don't let a discussion of negative area become a distraction.

MULTIPLICATION AS SCALING

Unlike the other models, the numbers don't need to be integers, so this page shows a continuous model of multiplication.

CONCLUSION

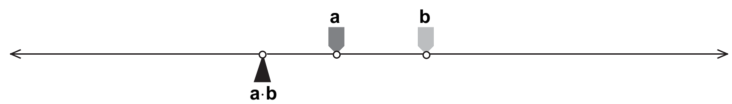
Finish the class discussion by asking students to compare the various models, particularly with regard to how they show the product of two negative numbers.

Mystery Machines

After Raz had been building his multiplying machines for a while, he decided to diversify. First he built addition, subtraction, and division machines, and then he decided to move from the arithmetic market into the more lucrative game market by building mystery machines. In this activity, you will explore some of his machines.

CHALLENGES

1. Open **Mystery Machines.gsp.**



The first machine is an ordinary multiplying machine, but with the numbers left off. Follow the directions in the sketch to find the locations of 0 and 1. Use the *New problem* button to try the challenge several times.

Q1 Describe the strategies you used to find the locations of 0 and 1.

2. On the second page (Machine 2), the challenge is to find and mark the location of the number 1/8. Repeat this challenge several times.

Q2 Describe your strategy clearly.

3. The third page (Machine 3) takes two numbers, a and b, and computes their sum, $a + b$.

Q3 Can you find 0? If so, describe your strategy; if not, explain why not. Can you find 1? If so, describe your strategy; if not, explain why not.

Look for clues to help you determine what rule is being used to calculate *c*.

4. Each Combo page contains a new machine that takes two numbers, a and b, and uses them to compute a third number, c. The machine uses a formula like $c = a + b + 2$, or $c = b - a$, or $c = 2a + b$. Investigate each machine by dragging a and b and observing the effects on c.

Q4 For each of the three pages, tell what formula is being used, and describe your strategy for discovering the formula.

EXPLORE MORE

Raz's Combo machines can give you some ideas for inventing your own machine.

Q5 The Build It Yourself page gives directions for building your own math machine. Follow the directions to build a division machine. Then change the calculation to build a machine of your own invention.

Objective: Students explore several mystery machines built on number lines that are missing their numbers. By dragging and observing markers representing a multiplication problem or an addition problem, students determine the locations of hidden numbers. Students manipulate input markers on other machines to determine the arithmetic operation performed by the machine.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should do another activity involving number machines (such as Raz's Magic Multiplying Machine) before undertaking this one.

Sketchpad Level: Easy. Students manipulate pre-made sketches. The Explore More section gives detailed directions for students to build their own machines. This is considerably more difficult.

Activity Time: 30–50 minutes. There's lots of flexibility in the amount of time students spend exploring the open-ended questions in the Explore More section. The Challenges section and the Explore More section can be done separately.

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Mystery Machines.gsp** in either setting)

CHALLENGES

Q1 To find 0, drag a (or b) until it and $a \cdot b$ are right on top of each other. This spot is 0. This method works because any number times 0 equals 0. Thus, when a is 0, $a \cdot b$ will also be 0. Once you've positioned a, you can check your work by dragging b. If a is at zero, then a and $a \cdot b$ will remain on top of each other as b is dragged.

To find 1, drag a until b and $a \cdot b$ are right on top of each other (or drag b until a and $a \cdot b$ are right on top of each other). a is at 1. This method works because any number times 1 equals itself. Thus, when a is 1, b and $a \cdot b$ will equal each other. Again, check your work by dragging. If a is 1, then b and $a \cdot b$ will remain on top of each other as b is dragged.

Q2 Move a and b so that they're both at 1/2. $(1/2)(1/2) = 1/4$, so $a \cdot b$ is at 1/4. Mark this spot using one of the gold arrows. Now move either a or b to 1/4 (where you just marked). $a \cdot b$ will now be at 1/8 because $(1/2)(1/4) = 1/8$.

Q3 To find 0, drag a until b and $a + b$ are right on top of each other (or drag b until a and $a + b$ are right on top of each other). This works because any number plus 0 equals itself. Thus, when a is 0, b and $a + b$ will equal each other. (Note that the method for finding 0 on an "adding machine" is the same as that for finding 1 on a "multiplying machine." This is because 1 is the *identity element* for multiplication and 0 is the *identity element* for addition.)

It's impossible to find 1, because there's no way to find the scale using addition.

Q4 Mystery combo 1: $c = a + 2b$

Mystery combo 2: $c = 4a - b$

Mystery combo 3: $c = ab - 1$

EXPLORE MORE

Q5 This challenge involves creating an algebraic construction. This is somewhat easier to build than the geometric construction, and the algebraic approach allows the machine to perform much more complex computations.

WHOLE-CLASS PRESENTATION

Use the Challenges in this activity to encourage a vibrant class discussion. As the class looks at each challenge, have students take turns operating the computer while other students make suggestions about how to investigate each machine. The challenge posed by Machine 3 is particularly interesting because one of the questions has no solution: It's not possible to find the position of 1. (Sometimes a result of impossibility is more enlightening than a more straightforward challenge with a well-defined answer.)

When students work with the Combo pages, be sure to have them explain and defend their strategies. Emphasize how important it is that they be able to explain their thought processes, and point out that they don't fully understand a problem until they can explain its solution to someone else.

The Explore More section is not particularly suited for use in a whole-class presentation.

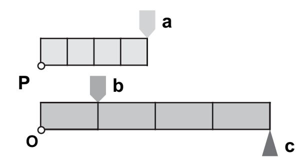
Dividing Real Numbers

When you first learned about division, your teacher probably began with a problem about distributing something fairly, such as "Divide 12 marbles among three children." Problems like this go only so far, because marbles and children come in positive integers. Division has to work for negative numbers and fractions.

MULTIPLICATION MACHINE

To understand division, you must first understand multiplication, so this activity starts with a multiplication machine.

1. Open **Division Machine.gsp.**



Q1 Experiment by dragging markers *a* and *b*. How can you control the number of rectangles in the blue bar? How can you control their width?

Q2 How do *b* and *c* behave when *a* is set to show exactly one yellow square?

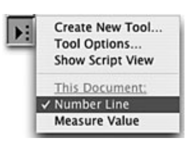
To do real multiplication and division, you should have numbers, and for that you need a number line. You can follow steps 2–6 below to make your own number lines, or you can go to page 2 of the sketch and skip to Q3 on the next page.

To merge the points, use the **Arrow** tool to select both of them and then choose **Edit | Merge Points.**

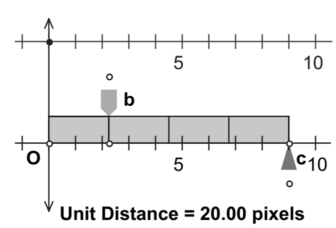
2. Press the *Show Number Line* button. Attach the blue bar to the number line by merging point *O* on the blue bar and point *O* on the number line.

You'll need a separate number line for the yellow bar so that the bars don't overlap.

3. Construct a line through point *O*, perpendicular to the existing number line, by using the **Arrow** tool to select point *O* and the line and then choosing **Construct | Perpendicular Line.**



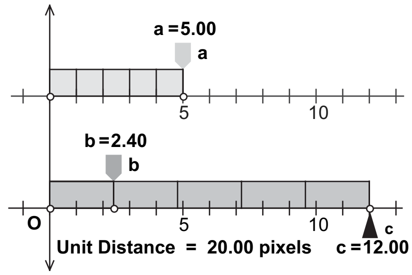
4. Use the **Number Line** custom tool to create the second number line. Press and hold the **Custom** tools icon, and choose **Number Line** from the menu that appears. Use the tool by clicking twice: first on the *Unit Distance* measurement and then on the perpendicular at the position where you want the new number line to appear.



5. Choose the **Arrow** tool, and attach the yellow bar to the new number line by merging point *P* with the zero point of the number line.

To hide points, use the **Arrow** tool to select them, and then choose **Display | Hide Points.**

6. To measure the position of each marker on the number line, use the **Measure Value** custom tool. Click each of the three points that appear above or below markers *a, b,* and *c.* Hide the three points.



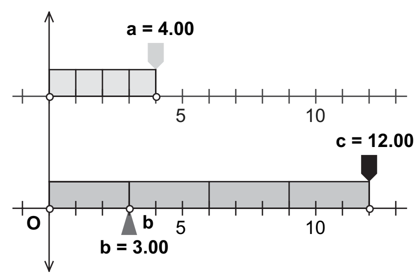
Q3 If you know the length of one blue tile and the number of tiles, you should be able to calculate the length of the row of blue tiles. How would you do this calculation? Write your answer as an equation using *a, b,* and *c.*

Choose **Measure | Calculate.** Click on the measurements to enter them in the calculation.

7. Use the measurements of *a* and *b* to calculate $a \cdot b$. Drag *a* and *b* to make sure that your calculation is correct.

DIVISION MACHINE

8. Press the *Multiply/Divide Toggle* button.



Q4 What changed when you pressed the button? Which markers can you control now? How is the behavior of the machine different?

If you can't get the exact numbers you want, click the *Round* button for the value you're moving, and then try again.

Q5 By construction, $c = a \cdot b$. if you know the values of *a* and *c*, how would you use those numbers to calculate *b*?

Q6 Use the division machine to calculate $15 \div 3$. (To do this, drag *c* to 15 and drag *a* to 3.) What is *b*? Then use the same method to calculate $7.2 \div 2.4$.

Q7 Use your division machine to answer the following questions. For each question, tell how you dragged markers *a* and *c* to investigate, describe your observations, and explain your answers.

a. What is the result when you divide a number by one?

b. If you divide by a negative number, is the answer always negative?

c. If you divide a positive number, is the answer always a smaller number?

d. What happens when you try to divide by zero?

Dividing Real Numbers

Objective: Students run a multiplication machine in reverse to perform division problems, to observe the relationship between multiplication and division, and to investigate properties of division.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None. This is a review topic for most Algebra 1 students, though it presents things in a new way.

Sketchpad Level: Intermediate. Students perform several constructions and use two custom tools. The activity includes explicit instructions for performing these tasks. Students can use page 2 to skip the construction steps so that they only need to drag objects and press buttons.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Division Machine.gsp** in either setting)

MULTIPLICATION MACHINE

Q1 Marker *a* determines how many rectangles appear in the blue bar. Marker *b* determines the width of each blue rectangle.

Q2 When *a* shows exactly one yellow square, markers *b* and *c* move in unison.

Q3 You can calculate the length of the row of blue tiles by multiplying the number of tiles by the length of each tile. As an equation, $c = a \cdot b$.

DIVISION MACHINE

8. This is the simplest step of the activity, but in many ways the most important. Encourage students to observe closely and to think about what they observe.

Q4 When students press the button, the control for *b* goes below the line and the control for *c* goes above it, and students find that they can now control *c* rather than *b*. This is what makes it a division machine now: It calculates $b = c \div$ a.

Geometrically, students are now controlling the size of the blue rectangles by dragging the end of the bar (representing the product) rather than by dragging the end of the first rectangle (the multiplier).

Q5 You would divide *c* by *a*. $b = c \div a$

Q6 $b = 5$ $(15 \div 3 = 5)$

$b = 3$ $(7.2 \div 2.4 = 3)$

Q7 Emphasize to students the importance of these four questions in developing and demonstrating their skills of translation and interpretation. Students must translate the questions, which are in mathematical terms, into the terms of the model and the behavior of the markers; must manipulate the model appropriately; and then must interpret the behavior they observe, expressing their conclusions mathematically.

a. When you drag *a* to exactly one, markers *b* and *c* are at the same value, no matter where you drag *c*. Dividing by one does not change the number.

b. No. When you drag *a* to be negative (for example, -2) and also drag *c* to be negative (for example, -6), the result is positive. As an equation, $-6 \div (-2) = 3$. Dividing by a negative number always changes the sign, so dividing a negative by another negative gives a positive answer.

c. No. When you drag *a* between zero and one, the result is that *b* is larger than *c*. This means that dividing a positive number by a number between zero and one results in a larger number.

d. When you drag *a* toward zero while *b* is positive, the marker for *b* moves off the screen, showing a larger and larger result. When you get *a* to exactly zero, the machine breaks and the entire blue bar disappears. If you try the same thing while *c* is already zero, *b* also stays at zero as *a* gets close to zero, but when *a* is exactly zero, the machine breaks and *b* disappears.

Algebraically, if $c \div 0 = b$, then $c = 0 \cdot b$. If *c* is not zero, then there is no number *b* that will satisfy the second equation, because $0 \cdot b = 0$. On the other hand, if $c = 0$, then any real number will do for *b*, so there is no unique answer.

WHOLE-CLASS PRESENTATION

To present this activity to the whole class, start on page 2 of the sketch **Division Machine.gsp** and drag markers *a* and *b* to see how the machine performs multiplication. Ask students to explain the role of each variable in determining how the machine works. It's helpful to relate the behavior of this machine to the grouping model in the Multiple Models of Multiplication activity.

Once students are satisfied with the multiplying machine, press the *Multiply/Divide Toggle* button and observe the changes in the sketch. Switch this toggle several times to make sure students have seen the changes clearly, and then observe the behavior of the division machine's markers. As the class explores the parts of Q7, it may be helpful to switch back and forth several times between multiplication and division.

The Commutative Property

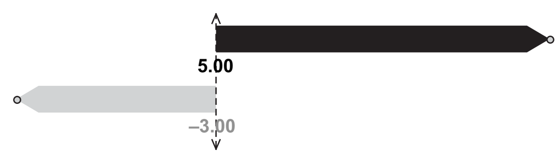
At a dinner table, you might ask for the salt and pepper, but you could ask for the pepper and salt and expect the same outcome. But sometimes the order of things is important. You can put on shoes and socks, but you'd better start with the socks.

An operation that is commutative is said to have the *commutative property.*

In each of the four elementary arithmetic operations, there are three numbers: the two you start with (called *operands*) and the result. Does it matter which operand comes first? If changing the order of the operands doesn't change the result, the operation is *commutative.*

ADDITION AND SUBTRACTION

1. Open **Commutative Property.gsp.** The top blue and green arrows are the operands. Drag the points at their tips to change their values.

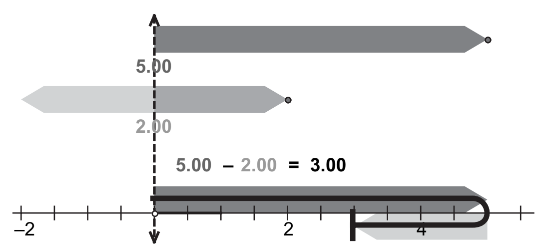


The top number line shows the addition in the order *blue* + *green.* The bottom number line shows the addition in the opposite order: *green* + *blue.*

For addition to have the commutative property, the two results must always be the same.

Q1 Are the results of the two addition problems (*blue* + *green* and *green* + *blue*) ever the same? Are they always the same? Drag the points to try many combinations of values. Does addition have the commutative property?

2. Go to the Subtraction page. Notice that in actually doing the subtraction problems, the bottom arrow has been flipped in order to subtract the second operand from the first.



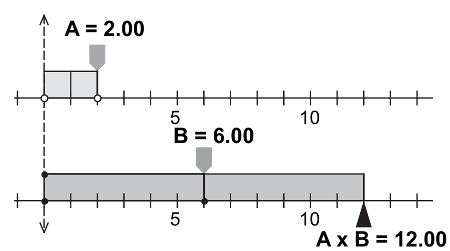
Q2 Experiment by changing the values. When is *blue* − *green* equal to *green* − *blue*? Are they always equal? Is subtraction commutative?

MULTIPLICATION AND DIVISION

You may have seen this model in a previous activity.

3. Go to the Multiplication page.

Q3 How does each blue bar correspond to a multiplication problem? Make a copy of the table below and finish filling in the first two columns for the blue bars.



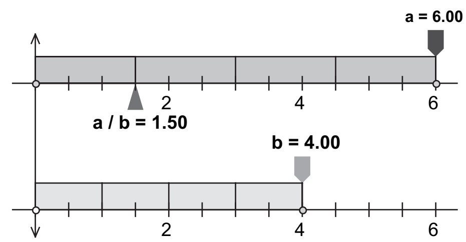
The entries in the first row indicate that marker *a* corresponds to the number of rectangles in the upper blue bar, and that its initial value is 2.

	Marker	Value 1	Value 2	Value 3
Number of upper blue rectangles	*a*	2		
Length of each upper blue rectangle				
Total length of upper blue bar				
Number of lower blue rectangles	*b*	6		
Length of each lower blue rectangle				
Total length of lower blue bar				

Q4 Create two more problems by moving markers *a* and *b*. Use your results to fill in the remaining columns of the table. When do the two red markers (showing the lengths of the two blue bars) represent equal products? Are they always equal? Is multiplication commutative?

4. Go to the Division page.

Q5 How does each blue bar correspond to a division problem? Copy this table and finish filling in the first two columns.



	Marker	Value 1	Value 2	Value 3
Total length of upper blue bar	*a*	6		
Number of upper blue rectangles				
Length of each upper blue rectangle				
Total length of lower blue bar	*b*	4		
Number of lower blue rectangles				
Length of each lower blue rectangle				

Q6 Create two more problems by moving markers *a* and *b*. Use your results to fill in the remaining columns of the table. When do the two red markers (showing the lengths of the individual rectangles in the upper and lower blue bars) represent equal quotients? Are they always equal? Is division commutative?

ALL TOGETHER

5. Go to the Summary page. The red bars represent the operands, *a* and *b*. Experiment with different values by dragging the points at their tips.

Q7 What do the blue bars represent? Why are there two of them?

Be sure to drag each operand separately. Don't drag them both at the same time.

Q8 Drag each operand in turn to change its value. Which of the four operations are commutative? How can you tell this from the behavior of the bars?

The Commutative Property

Objective: Students use dynamic models of the four arithmetic operations to determine which of the operations are commutative.

Student Audience: Pre-algebra/Algebra 1/Algebra 2

Prerequisites: None

Sketchpad Level: Easy. Students manipulate a prepared sketch.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Commutative Property.gsp** in either setting)

This activity and the one on the associative property provide an enjoyable way for students to investigate a bit of algebra theory.

ADDITION AND SUBTRACTION

Q1 The results of the two addition problems are always the same. Therefore addition is commutative.

Q2 The results of the two subtraction problems are the same only when the two operands are equal; under any other circumstances, the results are different. Subtraction is *not* commutative. The results $a - b$ and $b - a$ are opposite.

MULTIPLICATION AND DIVISION

Q3 The length of the blue bar represents a multiplication problem in which the length of an individual rectangle is multiplied by the number of rectangles to get the total bar length. Following is the table filled in. (Answers will vary in the last two columns.)

	Marker	Value 1	Value 2	Value 3
Number of upper blue rectangles	a	2	2	−2
Length of each upper blue rectangle	b	6	5	3
Total length of upper blue bar	$a \cdot b$	12	10	−6
Number of lower blue rectangles	b	6	5	3
Length of each lower blue rectangle	a	2	2	–2
Total length of lower blue bar	$b \cdot a$	12	10	–6

Q4 No matter how the operands are changed, the two multiplication problems give the same answer. Multiplication is commutative.

Q5 The length of the blue bar represents a division problem in which the entire length (the dividend) is divided into a number of equal pieces. The number of pieces is the divisor, and the length of each individual piece is the quotient. Following is the table filled in. (Answers will vary in the last two columns.)

	Marker	Value 1	Value 2	Value 3
Total length of upper blue bar	a	6	−8	5
Number of upper blue rectangles	b	4	4	5
Length of each upper blue rectangle	a/b	1.5	−2	1
Total length of lower blue bar	b	4	4	5
Number of lower blue rectangles	a	6	−8	5
Length of each lower blue rectangle	b/a	0.67	−0.5	1

Q6 The two red markers (the quotients of the two division problems) represent equal quotients only when the absolute values of the dividend and divisor are equal. For instance, if a and b are both 3, then $a/b = b/a = 1$, or if $a = 4$ and $b = -4$, then $a/b = b/a = -1$. Because the results are not always equal, division is not commutative.

ALL TOGETHER

Q7 The blue bars represent the result of adding the two operands a and b. The first blue bar shows the result for $a + b$, and the second one shows the result for $b + a$.

Q8 As the operands are dragged, the two bars for addition and multiplication are always equal in length, indicating that these two operations are commutative. The green bars for the two subtraction results move in opposite directions, as do the magenta bars for division. Thus subtraction and division are not commutative.

WHOLE-CLASS PRESENTATION

Use the pages of **Commutative Property.gsp** to investigate whether each of the arithmetic operations is commutative. The Multiplication and Division pages are easier to understand if students are familiar with the grouping models presented in Multiple Models of Multiplication and in Dividing Real Numbers. Get a number of students to formulate their conclusions in their own words to make sure that all of them understand what commutativity means and why some operations are commutative and others are not.

The Associative Property

Imagine that you've bought three items at a store and that you're adding up the three prices (call them a, b, and c) to check the bill. Does it matter whether you first add a and b, and then add c to the result, or could you instead add b and c first, and then add the result to a? Does $(a + b) + c = a + (b + c)$?

Instead of adding, what if you were subtracting the three numbers? Does $(a - b) - c = a - (b - c)$? What if you were multiplying or dividing them? Would it matter in these cases whether you performed the first calculation on a and b, or whether you performed it on b and c?

To write the question so it applies to any of the four arithmetic operations, use $\otimes$ to stand for the operator. You want to know if you'll get the same result by doing $a \otimes b$ first as you will if you do $b \otimes c$ first. Using parentheses to show the order, you want to know if

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

Another way of phrasing the question is to ask whether the b should be *associated* with the a or the c. If you get the same answer either way, the operation is called *associative* and is said to have the *associative property.* In this activity you will investigate this question for addition, subtraction, multiplication, and division to determine which of these operations are associative and which are not.

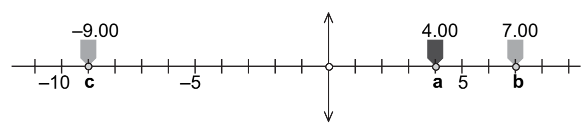
ADDITION

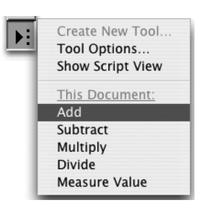
Begin by checking whether addition has the associative property. You must decide whether the following equation is true for all real numbers a, b, and c:

$$(a + b) + c = a + (b + c)$$

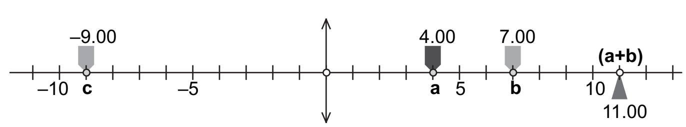
1. Open **Associative Property.gsp.**

There are two number lines with markers. For now, work on the top line.

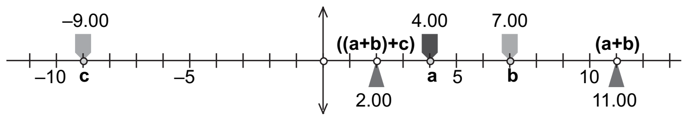




2. Press and hold the **Custom** tools icon to display the Custom Tools menu. Choose the **Add** tool. Click in order on points a and b. (You can use the **Text** tool to center the label of $(a+b)$ above its marker.)

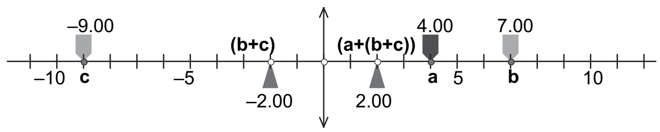


3. With the **Add** tool still active, click in order on point $(a+b)$ and point c.



Choosing the **Selection Arrow** tool deactivates the **Add** tool.

4. Use the **Arrow** tool to drag points *a*, *b*, and *c* and confirm that the new markers, representing $(a + b)$ and $(a + b) + c$, show the correct values.
5. On the lower number line, use the **Add** tool to construct $(b + c)$. Then construct $(a + (b + c))$.



You can drag the two number lines closer together for a better comparison. Change the scale by dragging the tick-mark numbers on L_1.

Q1 Compare the final values created on the two number lines. Drag *a*, *b*, and *c* to try different values. Do $(a + b) + c$ and $a + (b + c)$ always have the same value?

Q2 Does addition have the associative property? Explain in your own words what this means.

OTHER OPERATIONS

Finish the investigation on your own, using the Subtraction, Multiplication, and Division pages. There is a custom tool corresponding to each of the operations. In your investigation, decide which of these equations are true:

$$(a - b) - c = a - (b - c)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$(a \div b) \div c = a \div (b \div c)$$

Q3 Of the four elementary arithmetic operators, which have the associative property and which do not? Illustrate each conclusion with an example.

EXPLORE MORE

In each section of this investigation, you had to try many different values for the given points. Even if an operation does not have the associative property, there may be some special combination of values for *a*, *b*, and *c* for which the equation is true.

Q4 Go back to the operations that are not associative. Using your construction, find a case in which the equation is true even though there is no associative property.

The Associative Property

Objective: For each elementary arithmetic operation, students experiment with values on the number line to see if the operation is associative.

Student Audience: Pre-algebra/Algebra 1/Algebra 2

Prerequisites: Students should understand the number line representation of real numbers and the meaning of parentheses in the order of operations.

Sketchpad Level: Easy/Intermediate. Students use custom tools and may require a bit of extra guidance the first time. Otherwise, they simply drag objects.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Associative Property.gsp** in either setting)

This activity and the one on the commutative property provide an enjoyable way for students to investigate a bit of algebra theory.

ADDITION

2. Students first use a custom tool in step 2. If they are not experienced with custom tools, there may be a tendency to leave the tool activated and inadvertently click on the screen, creating unwanted objects. Tell them to choose the **Selection Arrow** as soon as they finish step 3.

Q1 The values $(a + b) + c$ and $a + (b + c)$ are equal for any a, b, and c.

Q2 Addition does have the associative property. Students' explanations will vary. One explanation that helps them remember the meaning of this property is the following: "When you add three numbers, it doesn't matter whether the second number associates with the first number or whether it associates with the third number. Whichever number it adds itself to, the final result will be the same."

OTHER OPERATIONS

Q3 Addition and multiplication have the associative property. Subtraction and division do not.

EXPLORE MORE

Q4 If students have trouble with this question, consider giving them a hint. In mathematics, special things tend to happen around the numbers zero and one.

For subtraction, even without associativity, the equation is satisfied when $c = 0$. For division, the equation is satisfied for any one of these conditions: $a = 0$, $c = 1$, or $c = -1$.

WHOLE-CLASS PRESENTATION

Use the pages of **Associative Property.gsp** to investigate whether each of the arithmetic operations is associative. The presentation requires the use of custom tools. If you're new to using custom tools, practice the steps described in the activity sheet before presenting to the class.

As you present this activity, emphasize that you follow a different order when calculating the upper answer and the lower answer, and then check whether the two answers are the same regardless of order. In the discussion it's particularly helpful to have a number of students describe in their own words what it is about the behavior of the markers that allows them to decide whether a particular operation is associative.

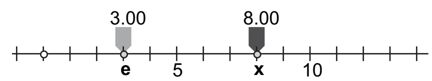
Identity Elements and Inverses

This activity introduces the concepts of identity elements and inverses. You'll determine whether addition, subtraction, multiplication, and division have identity elements. For any that do have identity elements, you'll investigate whether particular values have inverses.

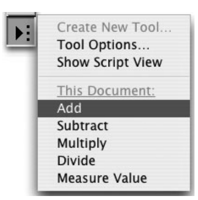
IDENTITY ELEMENT FOR ADDITION

An operation $\otimes$ has an identity element e if, for all possible values of x, $x \otimes e = e \otimes x = x$.

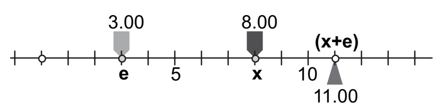
1. Open **Identity Elements.gsp.**



If an identity element e exists for addition, then $x + e = e + x = x$ for all possible values of x. On the upper number line, there are adjustable markers for x and e. You must create markers for $x + e$ and for $e + x$.



2. Press and hold the **Custom** tools icon to display the Custom Tools menu. Choose the **Add** tool. Click in order on points x and e.



3. With the **Add** tool still active, click in order on points e and x.

You can use the **Text** tool to separate the labels for (x+e) and (e+x) so they don't overlap.

Q1 Where does the $e + x$ marker appear relative to the $x + e$ marker? Does this relationship continue when you drag points x and e? What property of addition does this relationship represent?

So far, you have determined that $x + e = e + x$. But there's one more requirement for e to be the identity element for addition: Both of these quantities must be equal to x.

4. Drag e until all three quantities (x, $x + e$, and $e + x$) are equal.
5. Drag x to determine whether this identity relationship is true for all values of x. Be sure to try negative values and values near zero.

Q2 If $x + e = e + x = x$ is true for all values of x, then e is the identity element for addition. Does addition have an identity element, and if so what is it?

INVERSES FOR ADDITION

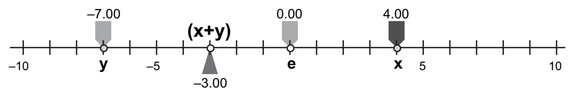
If an operation has an identity element *e*, then particular numbers may also have inverses with respect to that operation. An inverse of *x* with respect to ⊗ is defined as a value *y* such that $x \otimes y = y \otimes x = e$.

Addition has an identity element. Find out whether numbers have inverses with respect to addition. For any particular value of *x*, you're looking for a value *y* such that $x + y = y + x = e$.

Be sure marker *e* remains at the position of the identity element for addition.

6. On the lower number line, use the **Add** tool to add $x + y$, and use it again to add $y + x$.

Q3 Where do $x + y$ and $y + x$ come out in relation to each other? Why?



Q4 Use the **Arrow** tool to drag *y* so that $x + y = e$, and record the values of *x* and *y*. Then move *x* to a new position, drag *y* again so that $x + y = e$, and record these values. Repeat until you have four pairs of values for *x* and *y*. What can you conclude about inverses with respect to addition?

IDENTITY ELEMENT AND INVERSES FOR OTHER OPERATIONS

Q5 If a particular operation doesn't have an identity element, numbers cannot have inverses with respect to that operation. Why?

Q6 Repeat your investigation for other operations, using the remaining pages of **Identity Elements.gsp.** Record your results in a table like the one below. If an operation has no identity element, write "none" in the appropriate box.

Operator	Identity Element	Examples of Inverses
Addition		
Subtraction		
Multiplication		
Division		

Q7 For those operations with an identity element, are there any numbers that do not have inverses?

Objective: For each elementary arithmetic operation students determine whether the operation has an identity element, and if so, whether numbers have inverses with respect to that operation.

Student Audience: Pre-algebra/Algebra 1/Algebra 2

Prerequisites: Students should understand the number line representation of real numbers and should be familiar with the commutative property.

Sketchpad Level: Easy/Intermediate. Students use custom tools and may require a bit of extra guidance the first time. Otherwise, they simply drag objects.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Identity Elements.gsp** in either setting)

This activity and the ones on the commutative property and associative property provide an enjoyable way for students to investigate a bit of algebra theory.

This activity gives detailed directions for investigating the identity element and inverses for addition, and then has students perform their own investigations for the other operations.

IDENTITY ELEMENT FOR ADDITION

Q1 The $e + x$ marker appears at the same spot as the $x + e$ marker, no matter where you drag x or e. This occurs because addition is commutative.

Q2 Addition does have an identity element—the number 0.

INVERSES FOR ADDITION

Q3 The markers for $x + y$ and $y + x$ come out at the same place, because addition is commutative.

Q4 Recorded values of x and y will vary, but in each case, the numbers should have opposite signs and identical magnitudes.

IDENTITY ELEMENT AND INVERSES FOR OTHER OPERATIONS

Q5 If an operation doesn't have an identity element, there can be no inverses with respect to that operation, because the definition of an inverse depends on the value of the identity element.

Q6 Actual numbers will vary. Here's a typical table:

Operator	Identity Element	Examples of Inverses
Addition	0	$(3, -3), (-5, 5), (1.5, -1.5)$
Subtraction	none	none
Multiplication	1	$(2, 0.5), (-0.25, -4), (10, 0.1), (-3, -1/3)$
Division	none	none

Q7 The number zero has no inverse with respect to multiplication. You may want to ask students how their answer to this question will change if they limit themselves to the set of positive integers and zero, or to the full set of integers.

In the case of positive integers and zero, the number 0 is the only number that has an inverse with respect to addition, and the number 1 is the only number that has an inverse with respect to multiplication.

In the case of the full set of integers, every number has an inverse with respect to addition, but only the number 1 has an inverse with respect to multiplication.

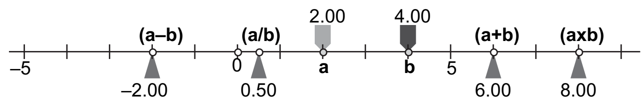
WHOLE-CLASS PRESENTATION

Use the pages of **Identity Elements.gsp** to investigate whether each of the arithmetic operations has an identity element and inverses. The presentation requires the use of custom tools, and is best done after you and your students have performed at least one other activity using custom tools.

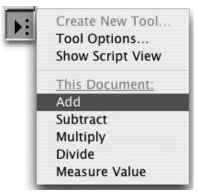
As you present this activity, have a different student operate the computer as you investigate each of the four operations. In the discussion of each of the four conclusions, it's particularly helpful to have several students describe in their own words why the behavior of the markers for a particular operation shows or does not show an identity element and inverses. Be sure to include Q7 in the discussion, so that students realize that some numbers may not have inverses even when there's an identity element.

Exploring Properties of Operations

In this activity you'll use Sketchpad arithmetic machines to investigate the properties of the four fundamental arithmetic operations.



INVESTIGATE



1. Open **Operation Properties.gsp.**
2. On the Addition page, use the **Add** custom tool to construct a marker showing the sum of *a* and *b.*
3. On a copy of the Operation Properties chart, make the title of the chart "Operation Properties for Addition," and fill in the blanks in the "Property" column with addition signs.
4. For each row of the chart, experiment by dragging *a* and *b* to determine whether that row's property is possible. If the property is not possible, write "Never" and explain why in the "When is it true?" column. If possible, fill in three examples, and write a sentence in the "When is it true?" column describing the conditions that must be met for the description to be true. Be specific about the values that the two numbers can have.
5. When you finish with one operation, go to the next page of the sketch, and use the appropriate custom tool to construct a result marker on that page. Then fill out the Operation Properties chart for that operation.

Here's an example of what you might write in the first row of the Addition chart:

Property	**Examples**	**When is it true?**
$a + b = 0$	$a = 5,\quad b = -5$ $a = -3,\quad b = 3$ $a = 0,\quad b = 0$	The sum of two numbers is zero when the numbers are the opposites of each other (or both equal zero).

PRESENT

You can use the Help system to learn how to make and use Movement, Animation, and Hide/Show buttons.

On the Presentation page of the document, choose one particular property of one of the operations, and create a presentation sketch that uses Movement, Animation, or Hide/Show buttons to demonstrate the circumstances in which the property is true.

Operation Properties for ____________________

Property	Examples	When is it true?
$a \square b = 0$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a \square b = 1$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a = b = a \square b$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a = a \square b$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a > 0$, $b > 0$, and $a \square b > 0$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a < 0$, $b < 0$, and $a \square b < 0$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a \square b > a$ and $a \square b > b$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a \square b$ is between a and b	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	
$a \square b < a$ and $a \square b < b$	$a =$ $b =$ $a =$ $b =$ $a =$ $b =$	

Geometric Descriptions of Algebraic Properties

You can use this chart to help you figure out what geometric features to look for when you investigate a particular algebraic property. For the algebraic property in each row of the table, write a sentence describing the corresponding geometric behavior of the markers for *a, b,* and $a \otimes b$. Row 7 is filled in as an example. ($\otimes$ stands for $+$, $-$, •, or $\div$.)

Row	Algebraic Property	Geometric Description
1	$a \otimes b = 0$	
2	$a \otimes b = 1$	
3	$a = b = a \otimes b$	
4	$a = a \otimes b$	
5	$a > 0$, $b > 0$, and $a \otimes b > 0$	
6	$a < 0$, $b < 0$, and $a \otimes b < 0$	
7	$a \otimes b > a$ and $a \otimes b > b$	The $a \otimes b$ marker is to the right of both the a and the b marker.
8	$a \otimes b$ is between a and b	
9	$a \otimes b < a$ and $a \otimes b < b$	

Objective: Students use arithmetic machines to explore various properties of the four fundamental arithmetic operations. They manipulate the machines by dragging two variables, *a* and *b*, observe the results of various calculations, and draw conclusions that reveal similarities and differences between the operations.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should have completed one of the other arithmetic machine activities. Experience with custom tools is helpful. The Present section requires students to create buttons.

Sketchpad Level: Intermediate. Students manipulate a pre-made sketch and use custom tools.

Activity Time: Will vary widely depending on the approach (see General Notes). It would take most students, working alone, much longer than one class period to fill in charts for all nine descriptions. By working in groups, students can complete the activity in one class period. Be sure to allow time for discussion and summarizing.

Setting: Paired Activity, or divide the class into groups of three or four (use **Operation Properties.gsp** in either setting)

GENERAL NOTES

This activity challenges students to analyze and understand the four fundamental arithmetic operations—addition, subtraction, multiplication, and division—from a visual perspective.

It's possible to answer any of the questions in this activity without using the arithmetic machines at all. But that defeats its purpose: Watching the machines in action as you drag pointers *a* and *b* can be fascinating. The four arithmetic operations, normally computed on a discrete, case-by-case basis, yield new insights when viewed from a continuous, motion-based perspective.

To get the most out of this activity, students should not treat the process of finding the answers as a purely mechanical one, dragging the pointers around without thinking about the meaning behind the various arrangements. Encourage students to discuss and reflect as they work.

Group Work: Dividing the class into groups will make the work more manageable and allow time for reflection. Each group can be responsible for filling out the four charts for addition, subtraction, multiplication, and division. Encourage group members to work together, observing each other's experiments and checking each other's work as they fill in the charts together, rather than having individual members work in isolation on specific charts.

Translating Between Representations: Doing this investigation requires students to translate between abstract algebraic statements and descriptions of concrete geometric behavior. First they must translate from the algebraic language in the chart ("$a \otimes b > a$ and $a \otimes b > b$") to the geometric behavior they are looking for ("The $a \otimes b$ marker is to the right of both *a* and *b*"). Once they have investigated the behavior of the model, they must then translate the geometric behavior they have observed back into algebraic language. This translation process is not easy for students, and it may be helpful to address the process explicitly, by having them fill out (perhaps as a class) the Geometric Descriptions of Algebraic Properties chart provided. Sample answers are provided following the Properties charts on the next pages.

An Alternate Approach: One possible variation to the approach suggested in the activity is to focus on one property for all four operations. For example, students might focus on the first property ($a \otimes b = 0$) and fill out the top row of each of the four charts—for addition, subtraction, multiplication, and division.

Additional Investigations: The questions in this activity are the tip of the iceberg when it comes to exploring the arithmetic machines. Why not challenge your students to write and share some of their own questions? Here are a few to consider:

- Describe the behavior of $a \div b$ as *b* is dragged back and forth through 0.
- If $a \cdot b$ equals 0, does knowing the value of *a* allow you to determine a unique value for *b*?
- If $a \div b$ equals 0, does knowing the value of *a* allow you to determine a unique value for *b*?
- When is $a - b$ greater (or less) than $a + b$?
- When is $a \cdot b$ greater (or less) than $a \div b$?

- Under what circumstances does $a + b = a - b$ regardless of where you drag a?
- Under what circumstances does $a \cdot b = a \div b$ regardless of where you drag a?

INVESTIGATE

Following are sample answers for each of the four charts. The specific examples and the "When is it true?" description will vary; many correct responses are possible.

Addition Properties

$a + b = 0$	$a=3$, $a=-2$, $a=0$,	$b=-3$ $b=2$ $b=0$	The sum of two numbers is zero when the numbers are the opposites of each other (or both equal zero).
$a + b = 1$	$a=0$, $a=3$, $a=-3$,	$b=1$ $b=-2$ $b=4$	The sum of two numbers is one when the second number is one more than the opposite of the first.
$a = b = a + b$	$a=0$,	$b=0$	The only way two numbers can both be equal to their sum is when both numbers are zero.
$a = a + b$	$a=5$, $a=-3$, $a=0$,	$b=0$ $b=0$ $b=0$	The sum of two numbers is equal to the first number only if the second number is zero.
$a > 0$, $b > 0$, and $a + b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	When both numbers are positive, their sum is always positive.
$a < 0$, $b < 0$, and $a + b < 0$	$a=-1$, $a=-0.5$, $a=-2$,	$b=-1$ $b=-1$ $b=-3$	When both numbers are negative, their sum is always negative.
$a + b > a$ and $a + b > b$	$a=1$, $a=0.1$, $a=3$,	$b=1$ $b=2$ $b=4$	The sum is greater than either number provided both numbers are positive.
$a + b$ is between a and b	$a=1$, $a=-1$, $a=-5$,	$b=-2$ $b=0.1$ $b=4$	If one number is positive and the other negative, the sum is between the two numbers.
$a + b < a$ and $a + b < b$	$a=-1$, $a=-1$, $a=-10$,	$b=-1$ $b=-10$ $b=-1$	The sum is less than either number if both numbers are negative.

Subtraction Properties

$a - b = 0$	$a=3$, $a=-2$, $a=0$,	$b=3$ $b=-2$ $b=0$	The difference of two numbers is zero when the numbers are equal.
$a - b = 1$	$a=2$, $a=-1$, $a=1$,	$b=1$ $b=-2$ $b=0$	The difference of two numbers is one when the first number is one more than the second.
$a = b = a - b$	$a=0$,	$b=0$	The only way two numbers can both be equal to their difference is when both numbers are zero.
$a = a - b$	$a=5$, $a=-3$, $a=0$,	$b=0$ $b=0$ $b=0$	The difference of two numbers is equal to the first number only if the second number is zero.
$a > 0$, $b > 0$, and $a - b > 0$	$a=1$, $a=1.1$, $a=5$,	$b=0.5$ $b=1$ $b=4$	When both numbers are positive, their difference is positive if the first is greater than the second.
$a < 0$, $b < 0$, and $a - b < 0$	$a=-2$, $a=-1.5$, $a=-4$,	$b=-1$ $b=-1$ $b=-3$	When both numbers are negative, their difference is negative if the first number is less than the second.
$a - b > a$ and $a - b > b$	$a=-1$, $a=-3$, $a=3$,	$b=-1$ $b=-2$ $b=-4$	The difference is greater than either number as long as the second number is negative and the first is greater than twice the second.
$a - b$ is between a and b	$a=5$, $a=-9$, $a=13$,	$b=2$ $b=-4$ $b=6$	If the second number is positive, the difference is between the two numbers when the first is greater than twice the second. If the second number is negative, the difference is between the two numbers when the first is less than twice the second.
$a - b < a$ and $a - b < b$	$a=1$, $a=3$, $a=7$,	$b=1$ $b=2$ $b=4$	The difference is less than either number if the second number is positive and the first is less than twice the second.

Multiplication Properties

$a \cdot b = 0$	$a=0$, $a=-2$, $a=0$,	$b=-3$ $b=0$ $b=0$	The product of two numbers is zero when at least one of the numbers is zero.
$a \cdot b = 1$	$a=1$, $a=3$, $a=1/4$,	$b=1$ $b=1/3$ $b=4$	The product of two numbers is one when the numbers are reciprocals.
$a = b = a \cdot b$	$a=0$, $a=1$,	$b=0$ $b=1$	The only way two numbers can both be equal to their product is when both numbers are zero or both numbers are one.
$a = a \cdot b$	$a=5$, $a=-3$, $a=0$,	$b=1$ $b=1$ $b=1$	The product of two numbers is equal to the first number only if the second number is one.
$a > 0$, $b > 0$, and $a \cdot b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	The product of two positive numbers is positive.
$a < 0$, $b < 0$, and $a \cdot b < 0$	Never		The product of two negative numbers is never negative.
$a \cdot b > a$ and $a \cdot b > b$	$a=1.1$, $a=-1$, $a=3$,	$b=1.1$ $b=-2$ $b=4$	The product is greater than either number either when both numbers are negative or when both numbers are greater than one.
$a \cdot b$ is between a and b	$a=1/2$, $a=1/2$, $a=4$, $a=-3$,	$b=-2$ $b=3$ $b=1/4$ $b=1/4$	The product is between the two numbers if one number is between zero and one and the other is either negative or greater than one.
$a \cdot b < a$ and $a \cdot b < b$	$a=0.5$, $a=-2$, $a=2$,	$b=0.2$ $b=1.5$ $b=-0.1$	The product is less than either number if both numbers are between zero and one, or if one number is negative and the other is greater than one.

Division Properties

$a \div b = 0$	$a=0$, $a=0$, $a=0$,	$b=-3$ $b=2$ $b=1/2$	The quotient of two numbers is zero when the numerator is zero and the denominator is not zero.
$a \div b = 1$	$a=1$, $a=-2$, $a=4$,	$b=1$ $b=-2$ $b=4$	The quotient of two numbers is one when the numbers are equal but not zero.
$a = b = a \div b$	$a=1$,	$b=1$	The only way two numbers can both be equal to their quotient is when both numbers are one.
$a = a \div b$	$a=5$, $a=-3$, $a=0$,	$b=1$ $b=1$ $b=1$	The quotient of two numbers is equal to the numerator only if the denominator is one.
$a > 0$, $b > 0$, and $a \div b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	The quotient of two positive numbers is always positive.
$a < 0$, $b < 0$, and $a \div b < 0$	Never		The quotient of two negative numbers is never negative.
$a \div b > a$ and $a \div b > b$	$a=-1$, $a=0.2$, $a=0.2$,	$b=-2$ $b=0.5$ $b=0.1$	The quotient is greater than either number if both numbers are negative, or if the denominator is between zero and one and the numerator is greater than the square of the denominator.
$a \div b$ is between a and b	$a=5$, $a=1/8$, $a=8$,	$b=2$ $b=1/2$ $b=-4$	The quotient is between the two numbers if the denominator is greater than one and the numerator is greater than the square of the denominator, or if the denominator is less than one and the numerator is between zero and the square of the denominator.
$a \div b < a$ and $a \div b < b$	$a=5$, $a=-1$, $a=3$,	$b=-2$ $b=1/2$ $b=2$	The quotient is less than either number if (a) the denominator is less than zero and the numerator is greater than the square of the denominator, or (b) the denominator is between zero and one and the numerator is less than zero, or (c) the denominator is greater than one and the numerator is between zero and the square of the denominator.

Geometric Descriptions of Algebraic Properties

If you have students fill in this chart, here are sample geometric descriptions of the algebraic properties:

Row	Algebraic Property	Geometric Description
1	$a \otimes b = 0$	The $a \otimes b$ marker points at 0.
2	$a \otimes b = 1$	The $a \otimes b$ marker points at 1.
3	$a = b = a \otimes b$	The *a*, *b*, and $a \otimes b$ markers all point to the same location.
4	$a = a \otimes b$	The *a* and $a \otimes b$ markers point to the same location.
5	$a > 0$, $b > 0$, and $a \otimes b > 0$	The *a*, *b*, and $a \otimes b$ markers are all to the right of 0.
6	$a < 0$, $b < 0$, and $a \otimes b < 0$	The *a*, *b*, and $a \otimes b$ markers are all to the left of 0.
7	$a \otimes b > a$ and $a \otimes b > b$	The $a \otimes b$ marker is to the right of both the *a* and the *b* markers.
8	$a \otimes b$ is between *a* and *b*	The $a \otimes b$ marker is between *a* and *b*. (Note that the algebraic description given here isn't very algebraic. The real algebraic description would be $a < a \otimes b < b$ or $b < a \otimes b < a$.)
9	$a \otimes b < a$ and $a \otimes b < b$	The $a \otimes b$ marker is to the left of both *a* and *b*.