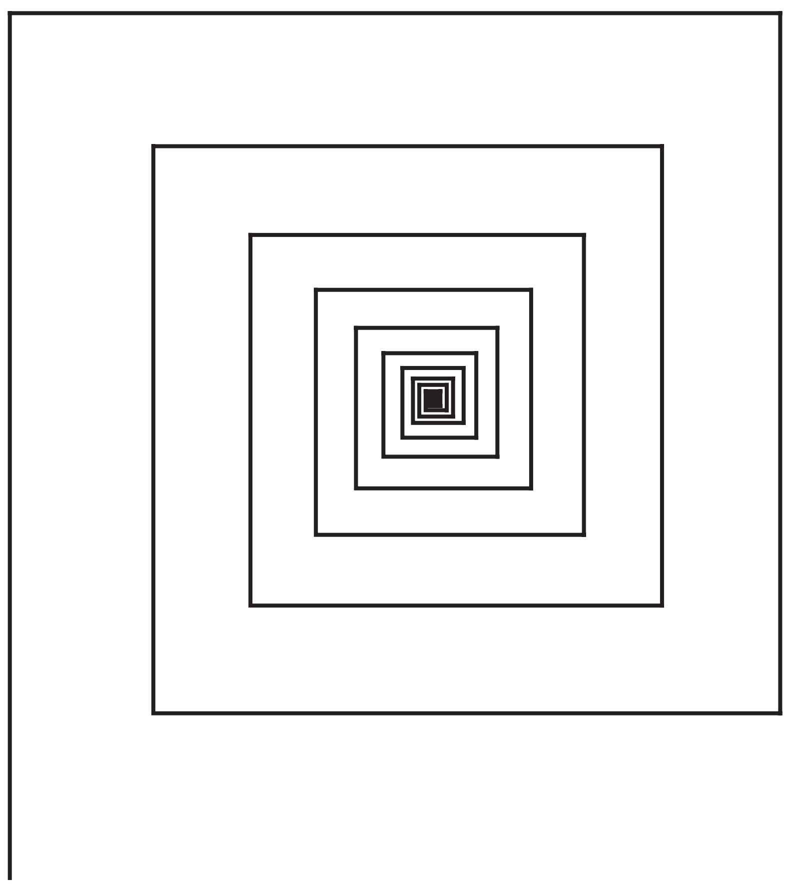
2

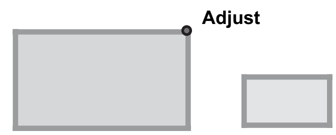
Ratios and Exponents



Ratio and Proportion

In this activity you will use a visual model of ratios and proportions in similar rectangles to solve problems involving proportions.

INVESTIGATE



1. Open **Proportion.gsp.** Drag point *Adjust* to see what it does.
2. A proportion requires two ratios. Show the first one by pressing the *Show Ratio* button. Then drag point *Adjust* again.

Q1 What does this ratio show?

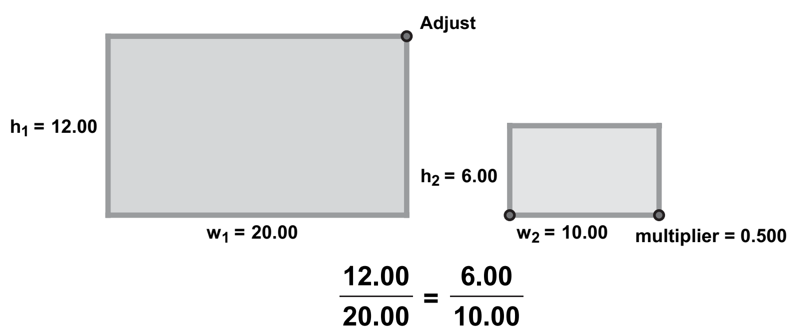
3. The second ratio will come from a second rectangle. Press the *Show Yellow Rectangle* button, and then drag *Adjust* again.

Q2 What happens to the yellow rectangle as you drag *Adjust*? What determines the shape and size of the yellow rectangle?

4. Press the *Show Multiplier* button and then drag point *Multiplier.*

Q3 What happens to the yellow rectangle?

Q4 How can you make each side of the yellow rectangle half as big as the corresponding side of the blue one?



5. To complete the proportion, press the *Show Second Ratio* button, and again drag points *Adjust* and *Multiplier.* Observe the effects on the proportion when you drag these two points.

The *Ratio of Integers* button can help you get the numbers exact.

6. Adjust the blue rectangle so its height-to-width ratio is 10.00/15.00. Press the *Set Multiplier to 1* button, and then make the yellow rectangle twice as big as the blue one horizontally and vertically.

Q5 What did you do to make the yellow rectangle twice as big as the blue one? What are the height and width of the yellow rectangle?

7. Adjust the blue rectangle so that h_1/w_1 is 8.00/10.00, and adjust the multiplier so that the yellow rectangle's height (h_2) is 6.00.

Q6 What multiplier did you use to do this? What is the resulting width (w_2) of the yellow rectangle?

Q7 In step 7, you actually solved the proportion $\frac{8.00}{10.00} = \frac{6.00}{w_2}$. Use the rectangles to solve the following proportions:

Show the size controls if you need to change the size of the rectangles.

$$\frac{8}{10} = \frac{12}{w} \qquad \frac{2.00}{3.00} = \frac{h}{7.50} \qquad \frac{x}{0.30} = \frac{1.70}{1.50} \qquad \frac{32}{m} = \frac{81}{45}$$

EXPLORE MORE

Go to page 2. This page lists eight different proportions using the height and width of the two rectangles. Some of the proportions are correct mathematically, but some are wrong. Determine which ratios are correct by manipulating the rectangles.

You can identify some incorrect proportions by making the rectangles square, and you can identify others by setting the multiplier to 1.

Q8 How can you tell which proportions are correct by manipulating the rectangles?

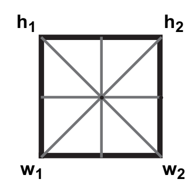
Q9 For each correct proportion, write down its letter, then the proportion as it appears using numbers, and finally the proportion using the symbols h_1, w_1, h_2, and w_2 in place of the numbers. For instance, proportion (a) is $\frac{10}{15} = \frac{20}{30}$, so you would write

$$(a) \quad \frac{10}{15} = \frac{20}{30} \qquad \frac{h_1}{w_1} = \frac{h_2}{w_2}$$

Choose **Measure | Calculate** to use Sketchpad's Calculator. Click on a measurement in the sketch to enter its value into the Calculator.

Q10 On page 3, calculate each of the 12 possible ratios involving h_1, w_1, h_2, and w_2. Arrange the ratios in pairs that have equal values. Write a proportion for each such pair. Which ratios do not belong to such pairs?

Go to page 4. This page shows the four variables from the rectangles at the four corners of a square. By reflecting the square across one of its axes of symmetry or by rotating by a multiple of 90°, you can generate all possible correct proportions. Use the square to check your answers for the proportions from pages 2 and 3.



Objective: Students manipulate two similar rectangles and study the proportions formed from the ratios of their sides.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: None

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Proportion.gsp**) or Whole-Class Presentation (use **Proportion Present.gsp**)

INVESTIGATE

Q1 The ratio shows a fraction representing the height of the rectangle divided by its width.

Q2 As you drag point *Adjust,* the yellow rectangle changes size and shape. It remains the same shape as the blue rectangle, and the same relative size.

Q3 As you drag point *Multiplier*, the size (but not the shape) of the yellow rectangle changes. When the multiplier is 1.000, the two rectangles are the same size.

Q4 You can drag point *Multiplier* until its value is 0.500. You can also look at the height and width measurements.

Q5 To make the yellow rectangle twice as big, drag point *Multiplier* until its value is 2.000. The height of the yellow rectangle is 20.00, and the width is 30.00.

Q6 To make the yellow rectangle's height 6 when the blue one's height is 8, you must use a multiplier of 0.750. The resulting width of the yellow rectangle is 7.50.

Q7 $w = 15$, $h = 5$, $x = 0.34$, and $m = 17.78$.

EXPLORE MORE

Q8 Answers will vary. One easy way to detect incorrect proportions is to set the multiplier to 1.00 and inspect the resulting fractions. Some of the incorrect ones (such as the one on the left below) can easily be identified.

$$\frac{10.00}{15.00} = \frac{15.00}{10.00} \qquad\qquad \frac{10.00}{15.00} = \frac{10.00}{15.00}$$

This method allows students to eliminate choices *b, c, e,* and *g.*

Another method is to adjust the multiplier so that the two ratios that make up a particular proportion have equal numerators, and then inspect the denominators. To test proportion *d,* you could manipulate the multiplier to be 1.50, causing *d* to appear as shown here. This proportion is obviously false because the two numerators match but the denominators don't.

$$\frac{15.00}{22.50} = \frac{15.00}{10.00}$$

The correct proportions are *a, f,* and *h.*

Q9 Here are the correct proportions expressed numerically and symbolically:

$$(a)\quad \frac{10}{15} = \frac{20}{30} \qquad \frac{h_1}{w_1} = \frac{h_2}{w_2}$$

$$(f)\quad \frac{20}{30} = \frac{10}{15} \qquad \frac{h_2}{w_2} = \frac{h_1}{w_1}$$

$$(h)\quad \frac{30}{20} = \frac{15}{10} \qquad \frac{w_2}{h_2} = \frac{w_1}{h_1}$$

You can use the square on page 4 to check the results on page 2. If you press the *Show Captions* button, you can edit the captions to change them to the corresponding numbers. The numbers will be displayed at the corners of the square, making it easy to check the proportions by transforming the square.

Q10 Here are the eight ratios that can be arranged in pairs:

$$\frac{h_1}{w_1} = \frac{h_2}{w_2} \qquad \frac{h_1}{h_2} = \frac{w_1}{w_2} \qquad \frac{w_1}{h_1} = \frac{w_2}{h_2} \qquad \frac{h_2}{h_1} = \frac{w_2}{w_1}$$

Here are the four ratios that cannot be arranged in pairs:

$$\frac{h_1}{w_2} \qquad \frac{w_1}{h_2} \qquad \frac{w_2}{h_1} \qquad \frac{h_2}{w_1}$$

WHOLE-CLASS PRESENTATION

In this presentation students will observe details of a model of a proportion using similar rectangles, manipulate the proportion by manipulating the model, use the model to solve various proportion problems, and investigate how any proportion can be written as an equation in eight different (but symmetrical) ways.

Begin by exploring the ratio of side lengths in a rectangle.

1. Open the sketch **Proportion Present.gsp** and drag point *Adjust.*

Q1 Ask, "What does dragging the point do to the rectangle?" After several answers, ask what two things about the rectangle are changed by moving point *Adjust,* and get a student to summarize that the point changes both the size and the shape of the rectangle.

2. Press the button labeled *Show Shape Buttons.* Press the shape buttons in turn to illustrate several different shapes.

3. Put the rectangle back into its original shape.

4. Press the *Show Ratio* button.

Q2 Ask, "What does the ratio represent?" Don't expect or impose any specific answers, but explore this question by changing the rectangle's shape.

Q3 Use either point *Adjust* or the shape buttons to make the rectangle tall and thin, and ask, "Is the ratio now a large number or a small one?"

Q4 Make the rectangle short and squat, and ask whether this ratio is a large number or a small one.

Q5 Ask, "What do you think will happen to the ratio if I press the *Square* button?" Press the button to test their conjectures.

Now add a second rectangle and look at its ratio.

5. Press the *Show Yellow Rectangle* button and drag point *Adjust.*

Q6 Ask, "What relationships do you see between the two rectangles as I drag *Adjust*?" Try to elicit student observations about both the relative shapes and the relative sizes.

6. Press the *Show Multiplier* button and drag point *Multiplier.*

Q7 Ask, "How does point *Multiplier* affect the relationship between the rectangles? How does it affect the relative shapes? How does it affect the relative sizes? What do you notice when the multiplier is 1.00? When it's 2.00?"

7. Set the multiplier back to 0.5. Press the *Show Second Ratio* button.

Q8 There's an equal sign between the ratios; ask, "Are the two ratios really equal?" Point out that the fact that they are equal means that the equation shown is a *proportion.*

8. Change the shape of the rectangles by dragging point *Adjust* or by pressing the shape buttons.

Q9 Ask, "Are the ratios still equal?" Ask, "How do you think the value of the multiplier is related to the numbers making up the two ratios?" Look at differently shaped rectangles to confirm students' conjectures.

Next use the rectangles to solve some proportion problems.

9. Go to page 2 and press the *Problem 1* button to display a proportion problem.

10. Have a student manipulate point *Adjust* so that the left side of the bottom proportion matches the left side of the problem. (When the point is close to the correct position, you can use the *Ratio of Integers* button to move it to the exact integer position.)

11. Have the student drag point *Multiplier* until the height of the yellow rectangle matches the number on the right side of the problem.

Q10 Ask, "Can you find the missing number in the problem by looking at the bottom proportion? What is the missing number?"

12. Problems 2, 3, and 4 on this page present additional challenges. To solve problems 3 and 4, show the size controls and press the smaller or larger buttons to make the scale of the rectangles appropriate for the numbers in the problem.

Explore how a particular proportion can be written in several different ways.

13. Go to page 3 and use the Calculator to compute various ratios. Make sure that students see that there are a number of different proportions that they can write using the same set of numbers.

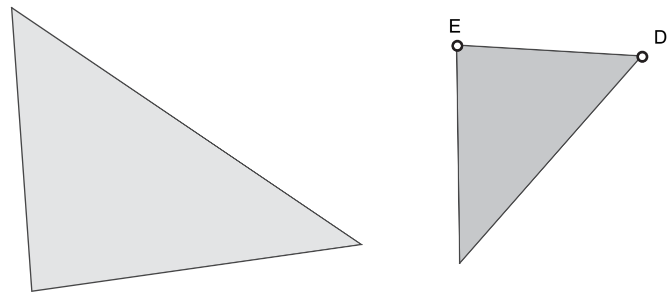
14. The square on page 4 presents an animation of the symmetry involved in the eight different ways of expressing the same relationship. Experiment with it to see the various ways of expressing the same proportion.

Proportions in Similar Triangles

Two geometric figures are *similar* if they have the same shape but not necessarily the same size. In this activity you'll investigate the properties of similar triangles and use proportions to find the missing sides of a pair of triangles.

INVESTIGATE

1. Open **Similar Triangles.gsp.**



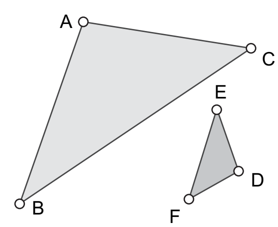
2. Drag points D and E to change the blue triangle.

Q1 As you drag the points, what changes and what stays the same?

Q2 Press the *Present Similarity* button. What does this seem to show about the two triangles?

3. To change the shape of the yellow triangle, you must first show its vertices. Press the *Show Vertices* button.

4. Drag the vertices of each triangle. Notice what changes and what stays the same.



Q3 Drag points D and E until $\triangle DEF$ exactly overlays $\triangle ABC$. Where do you have to place points D and E to make the triangles match up?

5. Drag the triangles apart again.

Q4 Make $\angle CAB$ in the yellow triangle very small. What happens in the blue triangle?

Q5 Drag point C far away from points A and B. What happens in the blue triangle?

6. Press the *Show Angle Measurements* button.

Q6 Drag each vertex to see what happens to the angle measurements. Which angle in the blue triangle corresponds to $\angle ABC$ in the yellow triangle? Which angle corresponds to $\angle BCA$? Which angle in the yellow triangle corresponds to $\angle FDE$?

Q7 What can you conclude about the corresponding angles of similar triangles?

To measure the length of a side, select it and choose **Measure | Length.** You may select all the sides and measure the lengths all at one time.

7. Measure all the sides of each triangle.
8. Calculate the ratio of one pair of corresponding sides. To do so, choose **Measure | Calculate,** click in the sketch on a length measurement from the yellow triangle, click the division symbol in the Calculator's keypad, and click on the corresponding length measurement from the blue triangle.

$m\,\overline{AB} = 6.4$ cm $\quad m\,\overline{ED} = 4.2$ cm
$m\,\overline{BC} = 7.6$ cm $\quad m\,\overline{EF} = 4.9$ cm $\quad \dfrac{m\,\overline{AB}}{m\,\overline{ED}} = 1.53$
$m\,\overline{CA} = 9.6$ cm $\quad m\,\overline{DF} = 6.3$ cm

9. Calculate the ratios of the other two pairs of corresponding sides.

Q8 What can you conclude about the ratios of the corresponding sides of these triangles?

Q9 Complete the following proportion using the ratios you calculated in step 8 and step 9. Then write two other proportions using these ratios.

$$\frac{m\overline{AB}}{} = \frac{}{m\overline{DF}}$$

Q10 Using your answers to Q7 and Q8, write a summary of the properties of similar triangles.

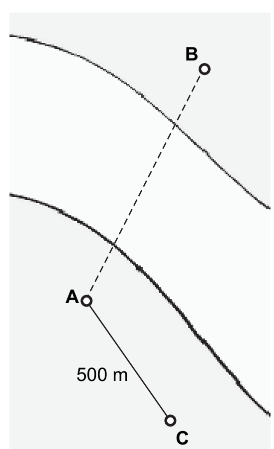
Q11 If $m\overline{AB} = 3.0$ cm, $m\overline{BC} = 4.0$ cm, and $m\overline{ED} = 5.0$ cm, use the triangles to find $m\overline{EF}$. Set up a proportion to check your answer.

Q12 Compare your answer to Q11 with other students' answers. Then compare your sketches. Explain any similarities or differences.

Q13 If $m\overline{CA} = 4.5$ cm, $m\overline{ED} = 2.0$ cm, and $m\overline{DF} = 3.5$ cm, use the triangles to find $m\overline{AB}$. Set up a proportion to check your answer.

EXPLORE MORE

Q14 Page 2 of the sketch shows a scale drawing of a river and the location of a bridge that must be built between point *A* and point *B*. There is no easy way to measure directly the distance between the two points on opposite sides of the river. Use similar triangles and proportions to find the required length of the bridge.



Proportions in Similar Triangles

Objective: Students manipulate two similar triangles, measure their angles and sides, and draw conclusions about the ratios of the sides. Students then use similar triangles to solve problems involving missing sides.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should already have been introduced to the terms *ratio* and *proportion.*

Sketchpad Level: Intermediate. Students measure the lengths of the sides of triangles and use the Calculator to compute ratios involving those lengths.

Activity Time: 20–30 minutes. This activity could be paired with another activity (such as Ratio and Proportion) in a single class period.

Setting: Paired/Individual Activity (use **Similar Triangles.gsp**) or Whole-Class Presentation (use **Similar Triangles Present.gsp**)

INVESTIGATE

Q1 As you drag points D and E, the angles of the blue triangle stay the same, but its size, position, and orientation change.

Q2 The *Present Similarity* button shows an animation that suggests the two triangles are the same shape.

Q3 The triangles match up when point D is on top of A and E is on top of B.

Q4 When you make $\angle BAC$ in the yellow triangle very small, $\angle EDF$ in the blue triangle also becomes small.

Q5 When you drag point C far away from points A and B, point F moves far away from points D and E.

Q6 Angle DEF in the blue triangle corresponds to $\angle ABC$ in the yellow triangle, and $\angle EFD$ corresponds to $\angle BCA$. Angle CAB in the yellow triangle corresponds to $\angle FDE$ in the blue triangle.

Q7 The corresponding angles of similar triangles are equal.

Q8 The ratios of the three pairs of corresponding sides of similar triangles are equal.

Q9 Students should write these three proportions or their equivalents:

$$\frac{m\overline{AB}}{m\overline{ED}} = \frac{m\overline{CA}}{m\overline{DF}} \qquad \frac{m\overline{AB}}{m\overline{ED}} = \frac{m\overline{BC}}{m\overline{EF}} \qquad \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{CA}}{m\overline{DF}}$$

Q10 Corresponding angles of similar triangles are equal, and the ratios of corresponding sides are equal.

Q11 $m\overline{EF} = 6.7$ cm. This result is rounded off to the nearest tenth, as are all the distances in this activity.

Q12 Because of rounding, some students may end up with slightly different answers such as $m\overline{EF} = 6.6$ cm.

Q13 $m\overline{AB} = 2.6$ cm. The proportion is

$$\frac{m\overline{AB}}{m\overline{ED}} = \frac{m\overline{CA}}{m\overline{DF}}$$

EXPLORE MORE

Q14 Although this problem is conceptually the same as the similar triangles in the activity, students may have trouble getting it. There are still two triangles. One triangle appears in the scale drawing, with sides that students should measure in cm. The other straddles the banks of the river itself, with one side of 500 m and another side that is the unknown length of the bridge. Students need to set up a proportion using the corresponding sides of these triangles. The bridge must be 888 m in length.

WHOLE-CLASS PRESENTATION

In this presentation students will observe the behavior of similar triangles, compare measurements of their sides and angles, find ratios that are equal, and use the ratios to write proportions. Students will then use the proportions to solve several different problems involving similar triangles.

To present this activity to the entire class, follow the Presenter Notes and use the sketch **Similar Triangles Present.gsp.**

1. Open **Similar Triangles Present.gsp.** Drag points *D* and *E.*

Q1 Ask, "What do you notice about the triangles as I drag points *D* and *E*?" The answers should include the fact that the position, orientation, and size change, and should also include the observation that the shape doesn't change or that the sizes of the angles don't change.

You may want to use the *Present in Reverse* button after the *Present Similarity* button.

2. Press the *Present Similarity* button. As the animation takes place, point out the three stages of the animation: translation (change of position); rotation (change of orientation); and dilation (change of size).

3. Show the vertices and drag points *A, B,* and *C.*

Q2 As you drag points *A, B,* and *C,* ask students to identify which vertex of the blue triangle corresponds to each vertex of the yellow one.

When the triangles are aligned, you can drag the blue one so it lies precisely on top of the yellow one.

4. Check the answers by clicking the *Align Triangles* button. With the triangles aligned, press the *Show Angle Measurements* button and the *Show Distance Measurements* button.

Q3 Ask students what they notice about the measurements. (Both sets of measurements are exactly equal between the two triangles.)

Q4 Drag the vertices and ask what students notice. (The angles remain equal, but the distances don't.)

Q5 Press *Show Ratios* and ask students what they observe about the ratios. (All three are equal.) Drag vertices to change relative sizes; students should observe that the ratios change, but all three remain equal to each other.

Tell students it's not fair to interchange the left and right sides of a proportion and count it as different.

Q6 Point out that proportions are made up of two equal ratios, and ask students to write down as many different proportions as they can from the three ratios. Press the *Show Proportions* button to verify the answers.

You can also use the buttons in the sketch to adjust the distances.

Q7 Go to page 2 of the sketch, and ask students to write and solve a proportion for the problem that appears. Then show the distance measurements and drag point *B* until $AB = 3.0$ cm, drag point *C* until $BC = 4.0$ cm, and drag point *E* until $ED = 5.0$ cm. Have students compare their results with distance *EF.*

5. Use a similar method to solve the problem listed on page 3 of the sketch.

6. On the Explore More page of the sketch, show the hint and let students read it.

7. Use the buttons to show a triangle, to show the distance measurements, and to make the triangle similar to triangle *ABC* on the map.

Q8 Ask students to set up a proportion using the two triangles and to find the length of the bridge. Use the buttons in the sketch to check their answers.

Rates and Ratios

You buy 52 ounces of detergent for 9 dollars. You run 100 meters in 15 seconds. You use 5 gallons of gas to drive 150 miles. Each of these pairs of numbers is an example of a *ratio*.

You can use ratios to answer questions like: If my pasta machine makes 7 noodles in 4 seconds, how many noodles will it make in 30 seconds? To answer such questions, you have to make some assumptions. In this activity, you will examine what those assumptions are and use them to answer questions like the ones posed here.

EXAMINE YOUR ASSUMPTIONS

This ratio is sometimes written as 52 : 9, and sometimes as $\frac{52}{9}$.

Suppose you can buy 52 ounces of detergent for 9 dollars. If you triple both the values in that ratio, you'll get 156 ounces for 27 dollars. The second ratio is considered equal to the first, because it consists of 3 copies of the original ratio.

Any pair of numbers, like the ones mentioned above, can be considered a ratio. Not all ratios, however, can be applied in the same way.

Q1 In your experience, if 52 ounces of detergent cost 9 dollars, will 156 ounces cost 27 dollars? Why or why not?

Q2 In your experience, if a person ran 100 meters in 15 seconds, will that person run 1000 meters in 150 seconds? Why or why not?

A *constant rate* is a rate that stays the same over time.

In both examples above, you can create a new ratio equal to the original ratio. Whether the new ratio is meaningful depends upon whether your ratio represents a *constant rate* (a rate that stays the same over time).

Imagine that each turn of the crank on your pasta machine takes a certain amount of time and produces a certain number of noodles.

1. Open **Rates and Ratios.gsp.**
2. To run the machine, press the *Turn the Crank* button.

Use the gold rectangles to count the noodles, and the blue ones to count the seconds.

Q3 How many noodles does the machine produce in one turn of the crank? How long does it take? Write the ratio of noodles to seconds as a fraction.

Q4 Turn the crank again. How many noodles do you have now? How many seconds has the machine been running?

Q5 Keep turning the crank until the machine has run for 32 seconds total. Without counting them, determine how many noodles the machine has made. How did you figure this out without counting?

Q6 Does the pasta machine run at a constant rate? Explain.

RATES AND PROPORTIONS

A *proportion* is a statement that two ratios are equal. You used the pasta machine to show the rate at which the machine runs in the form of two different ratios:

$$\frac{7\text{ noodles}}{4\text{ seconds}} = \frac{56\text{ noodles}}{32\text{ seconds}}$$

The second ratio extends the basic rate (7 noodles in 4 seconds) 8 times.

3. Go to page 2. This model of the pasta machine runs at the same rate but produces noodles continuously rather than in batches.

Since the machine runs at a constant rate, the rate for 4 seconds is the same as the rate for 30 seconds, and you can use the proportion

$$\frac{7\text{ noodles}}{4\text{ seconds}} = \frac{y\text{ noodles}}{30\text{ seconds}}$$

to determine how many noodles the machine can make in 30 seconds.

Model this by dragging the time slider—but don't count your noodles!

Q7 How many times do you need to repeat the rate "7 noodles in 4 seconds" to reach 30 seconds? Explain how to use this value to determine the value of y.

Q8 Describe a series of calculations that produces the value of y using the quantities 7 noodles, 4 seconds, and 30 seconds. Explain why this method works.

Q9 Describe the method you used in the previous question so that another person could use the method to solve any proportion.

Q10 If you repeat the ratio "7 noodles in 4 seconds" 15 times, you get 15 • 7 noodles in 15 • 4 seconds. That's 105 noodles in 60 seconds. If you repeat the ratio "y noodles in 30 seconds" 2 times, that's $2y$ noodles in 60 seconds. What is the value of y? Should it be the same as the value you found in Q7? Why?

Q11 If you've heard about "cross-multiplying" as a way to solve a proportion, explain why it works using the idea of "repeating" both ratios in a proportion.

EXPLORE MORE

Q12 Reduce the number of seconds to exactly 1 by dragging point *time*. How many noodles does the machine make in 1 second? How did you calculate this? Does multiplying this value by 30 give you the number of noodles made in 30 seconds? Why?

Q13 Describe the method you used in the previous question so that another person could use the method to solve any proportion.

Objective: Students explore a Sketchpad model involving rates and use proportions to explore rate problems.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: This activity is suitable for students with or without prior experience in solving proportions. Students should be able to solve equations of the form $ax = b$.

Sketchpad Level: Easy. Students manipulate points and change the values of parameters.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Rates and Ratios.gsp** in either setting)

Some textbooks describe "cross-multiplying" of proportions, and some avoid the technique. Q11 asks students to justify this technique. This question is optional, and you can tell students to skip it if you prefer.

EXAMINE YOUR ASSUMPTIONS

The model used in this activity is one way to see equal ratios. The visual model is designed to illustrate repeating a rate—every 4 seconds, you will see 7 more noodles. The model also marks off, with the red segments, groups of 7 noodles and 4 seconds, so that you can see that you have made 8 copies of the original ratio. The discrete nature of the machine makes it easier for students to see these copies. If students are actually counting noodles and seconds, they must count the colored intervals between the vertical line segments, not the segments themselves.

Students can change the rate at which their machine works by changing the rate parameters at the top of the sketch: the number of noodles and the number of seconds.

Q1 If you are buying 3 boxes of detergent, and each box has the same number of ounces and costs the same, then you can repeat the ratio in this way. However, students may be familiar with buying in bulk for a lower cost.

Q2 The runner may be able to keep up that same speed while running 10 times as far, but then again, she may not.

Q3 The machine produces 7 noodles in one turn of the crank, which takes 4 seconds. The ratio is

$$\frac{7 \text{ noodles}}{4 \text{ seconds}}$$

Q4 After two turns of the crank, there are 14 noodles, and the machine has been running for 8 seconds.

Q5 There are 8 copies of 4 seconds in 32 seconds, so there are 8 copies of 7 noodles, making 56 noodles.

Q6 The pasta machine produces the same number of noodles every 4 seconds and doesn't slow down or speed up over time. Some students will argue that this means it runs at a constant rate. Other students may point out that the machine doesn't produce any finished noodles until the end of the 4 seconds, so it's not really running at a constant rate—and that at 2 seconds, it has not produced even a single finished noodle. Both arguments have merit, and this question can lead to a very enlightening class discussion on what is meant by *constant rate,* and on the difference between discrete behavior (exemplified by the pasta machine on page 1) and continuous behavior (exemplified by the machine on page 2).

RATES AND PROPORTIONS

No matter how a proportion is solved, students should have a sense of the meaning of their calculations. Solving by finding the number of copies as in Q7 focuses students on the idea that they are repeating a rate.

Q7 You must repeat the 4-second ratio 7.5 times. You can see this in the model by thinking of 4 as a unit; there are 7.5 groups of 4 seconds in 30 seconds. The red segments shown in the model may help. Since you have 7.5 groups of 4 seconds, you will also have 7.5 groups of 7 noodles, or 52.5 noodles.

Q8 Calculate 30 seconds divided by 4 seconds times 7 noodles. Dividing 30 by 4 means 7.5 groups of 4 seconds. Since the rate is constant, there will be 7.5 groups of 7 noodles. Multiply 7.5 by 7.

Q9 Divide to find the number of times the ratio will be repeated. Multiply this number by the other quantity you are repeating.

Q10 Since 105 noodles equals $2y$ noodles, the value of y is 52.5. The two rates (y noodles in 30 seconds and 7 noodles in 4 seconds) are equal, so repeating both will result in ratios that are still equal. Whichever equation you use to solve for y, the value of y is the same.

Q11 Take the present question as an example. You would multiply 30 by 7, and 4 by y. The meaning of these steps is clearer if you write

$$\frac{30(7\text{ noodles})}{30(4\text{ seconds})} = \frac{4(y\text{ noodles})}{4(30\text{ seconds})}$$

This means that you have repeated the ratio on the left 30 times and the one on the right 4 times. Since each ratio gives the number of noodles made in 120 seconds, the numerators must be equal. Since $4y$ equals 210, y equals 52.5.

EXPLORE MORE

Q12 To calculate a unit rate equivalent to the given rate of 7 noodles in 4 seconds, divide both the numerator and denominator by 4. The unit rate is 1.75 noodles per second. Knowing how many noodles are made in a second allows you to multiply by 30 to find how many noodles were made in 30 seconds. As before, you are repeating a constant rate.

Q13 Divide to find a unit rate; multiply this unit rate by the number of units to which you are repeating the rate.

WHOLE-CLASS PRESENTATION

In this presentation students will observe a machine running at a certain rate and use a proportion to figure out how much it will produce during a certain period of time. Students will discuss what a constant rate means, why it's required to use a proportion, and how continuous and discrete processes differ.

Start with a Sketchpad model of a pasta machine producing noodles at a certain rate.

1. Open **Rates and Ratios.gsp.** Press the *Turn the Crank* button.

Q1 Ask, "How many noodles did the machine produce in one turn of the crank? How long did it take?" (Students can use the gold rectangles to count the noodles, and the blue ones to count the seconds.)

Q2 Ask students to write the ratio of noodles to seconds as a fraction.

Q3 Turn the crank again and ask students how many noodles there are now, and how many seconds the machine has been running.

2. Turn the crank 6 more times, so the machine has run for a total of 32 seconds.

Q4 Ask students to determine how many noodles the machine has made without counting them. Ask how they got their answers. Express the student explanations as proportions.

Discuss what a *constant rate* means.

Q6 Ask whether the pasta machine runs at a constant rate. This question does not have a clear-cut answer, and should lead to an interesting discussion about the meaning of constant rate and about discrete and continuous processes.

A slightly different machine can help students understand the idea of constant rate.

3. Go to page 2 and press the *Turn the Crank Once* button.

Q7 Ask students how this machine differs from the machine on page 1. Turn the crank again to help them observe the differences.

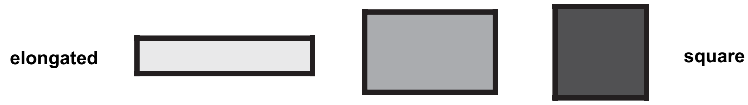
Q8 Ask students to figure out how many noodles this machine can produce in 30 seconds. They should do their calculation by setting up a proportion. Once they have their answers, check the answers by dragging point *time* to 30 seconds.

Have students summarize the results.

Q9 Ask students to describe the method they used to find the number of noodles so that another person could use the method to solve any proportion.

Q10 Ask students to explain why using proportions works only if the machine runs at a constant rate.

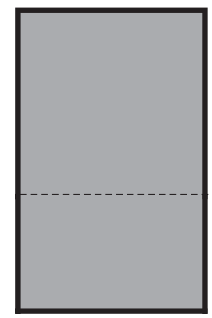
The Golden Rectangle and Ratio



The ratio of the width of a golden rectangle to its height is called the *golden ratio.*

You can describe the shape of a rectangle using the ratio of its width to its height. The rectangle on the left has a ratio of about 5:1, and the square on the right has a ratio of 1:1. The shape in the middle is often considered to be more attractive and has been called the *golden rectangle.* Paintings, photos, books, and magazines are often made with proportions similar to the golden rectangle.

If you add a square to the long side of a golden rectangle, the result is still a golden rectangle, oriented vertically rather than horizontally. The new, larger rectangle still has its sides in the same ratio as the original. You'll use this property to construct a golden rectangle, determine its ratio of width to height, and explore its characteristics.



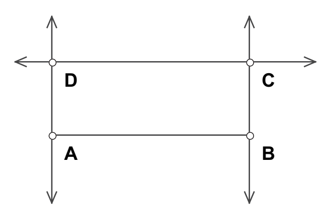
SKETCH AND INVESTIGATE

You'll begin by constructing an adjustable rectangle.

1. In a new sketch, use the **Segment** tool to draw segment *AB.*

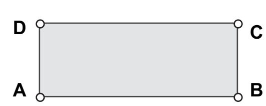
With the **Arrow** tool, select the segment and both points. Then choose **Construct | Perpendicular Lines.**

2. Construct two lines perpendicular to segment *AB,* one through point *A* and the other through point *B.*
3. Construct point *C* on the line through *B.*
4. Construct a line parallel to segment *AB* through point *C.* Construct intersection *D.*
5. Hide the lines and construct segments to connect the four points.



Select the points in order and choose **Construct | Quadrilateral Interior.**

6. Construct the quadrilateral interior.



Choose **Measure | Calculate** to show the Calculator. Click the measurements in the sketch to enter them into the Calculator.

7. Measure the lengths of segments *AB* and *AD* by selecting the segments and choosing **Measure | Length.** Calculate the ratio of the width to the height.

$m\,\overline{AB} = 4.48$ cm
$m\,\overline{AD} = 1.66$ cm
$\dfrac{m\,\overline{AB}}{m\,\overline{AD}} = 2.70$

Q1 What point must you drag to adjust the rectangle's shape? Drag that point to make the shape more attractive to you. What's the ratio now?

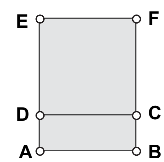
Now you'll add a square above the rectangle.

Use the Transform menu to mark a center and rotate a point.

8. Mark point *D* as the center of rotation, and rotate point *C* by 90° about *D*. Label the new point *E*.
9. Mark point *E* as the center of rotation, and rotate point *D* by 90°. Label the new point *F*.
10. You now have the four vertices of the added square. Construct the sides and interior of the square.

To measure the distance from *A* to *E*, select the two points and choose **Measure | Distance.**

11. The original rectangle and the new square together make a larger rectangle. Measure the longest side of this new rectangle. Then calculate the ratio of the longer and shorter sides of the large rectangle.



Q2 How does this ratio compare to the ratio from the original rectangle?

12. Adjust point *C* until the two ratios are as close to equal as you can make them.

The golden ratio is often represented by the Greek letter φ (phi).

Q3 What are the ratios now? This is the value of the golden ratio.

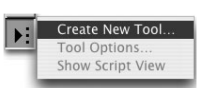
Q4 If you add another square on side *AE* to make a still larger rectangle, what do you think will be the ratio of the sides of this rectangle?

A GOLDEN SPIRAL

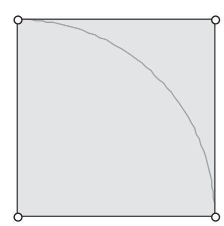
By constructing an arc inside square *CDFE* and then adding more squares with arcs, you can construct a golden spiral.

13. Select in order points *D, C,* and *E*. Choose **Construct | Arc On Circle.**
14. Hide the labels of points *C, D, F,* and *E*.

Press and hold the **Custom** tools icon to show the Custom Tools menu.



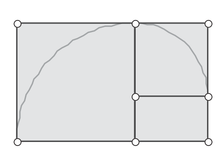
15. Create a tool to make it easy to repeat the process: Select points *C, D, F,* and *E,* the segments connecting them, the square interior, and the arc. Then choose **Create New Tool** from the Custom Tools menu, and name the new tool **Square With Arc.**



To use the new tool, choose it from the Custom Tools menu.

16. Use the new tool on points *A* and *E* to add a new square.

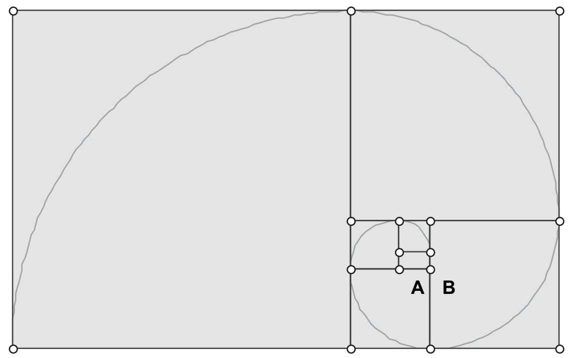
Q5 Measure the length and height of the new rectangle, and calculate their ratio. What result do you get?



To make more squares, you'll have to make the original shapes smaller.

Make sure each new arc connects to the previous arc.

17. Move *A* and *B* closer together, and use the new tool several times to add more squares to the existing rectangles.

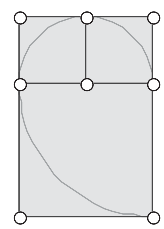


Q6 How many rectangles did you make?

Q7 What do you think is the ratio of the sides of the largest rectangle you made? Measure the distances and calculate the ratio to confirm your conjecture.

Q8 If your original rectangle was not a golden rectangle, what effect do you think there would be on the ratio of the sides of the largest rectangle? Drag point *C* to find out. What do you observe about the shape of the largest rectangle as you change the shape of the smallest one?

EXPLORE MORE



Q9 Start a new spiral by using the **Square With Arc** tool twice at the same size before you start adding larger squares. As the rectangles get larger, what happens to the ratio of the sides?

Q10 If the first square has sides of length 1, so does the second. How long are the sides of the third square? How long are the sides of the fourth square, and the fifth? Write down the sizes of the first 10 squares. Have you ever seen these numbers before?

Objective: Students use a ratio to describe the shape of a rectangle. They construct a golden spiral and examine the ratios of the rectangles in their construction.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should be familiar with ratios from other contexts.

Sketchpad Level: Challenging. Students use various tools; use commands from the Construct, Transform, and Measure menus; and create and use a custom tool. The instructions for all of these steps are reasonably detailed.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch needed) or Whole-Class Presentation (use **Golden Rectangle Present.gsp**)

SKETCH AND INVESTIGATE

If students are new to the tools, identify the **Segment** tool, the **Point** tool, the **Text** tool, and the **Custom** tools icon. Students must use the **Text** tool in several places in the activity in order to show, hide, or change point labels.

2. Students can construct the perpendiculars one at a time by selecting segment *AB* and one point before choosing the command, or they can construct both at once by selecting segment *AB* and both points before choosing the command.

4. To construct intersection *D,* students can use the **Point** tool or the **Arrow** tool, or they can select the two lines and choose **Construct | Intersection.**

6. This step assumes that students can find **Construct | Quadrilateral Interior** once they have selected the four points.

Q1 Drag point *C* to change the shape of the rectangle. Dragging points *A* and *B* changes the size of the rectangle but not its shape.

Q2 Answers will vary, depending on the shape of the original rectangle.

Q3 Students will not be able to get the two ratios exactly equal due to limitations of dragging points on the screen, but they should get an answer slightly greater than 1.6. When the ratios are equal, they are approximately 1.618. This is the golden ratio (φ).

Q4 Adding another square will produce a similar result to adding the first square: The new, larger rectangle will have the same shape as the smaller starting rectangle.

A GOLDEN SPIRAL

13. The **Arc On Circle** command is available because points *C* and *E* are equally distant from *D,* defining an implicit circle centered at *D.*

14. It's important to hide the labels so that the tool students will make in step 15 does not clutter the sketch with labels every time it is used.

15. The tool will work correctly only in one direction, so students will have to take care what point they click first. Fortunately, they can always use **Edit | Undo**.

Q5 The new, larger rectangle also has its width and height in the golden ratio.

Q6 Answers will vary, but it's best if students make at least four or five new rectangles.

Q7 Each new rectangle is golden, with its sides in the golden ratio.

Q8 Even if the original rectangle is not golden, successive rectangles become closer and closer to golden in shape.

EXPLORE MORE

Q9 As the rectangles get larger, the ratio becomes closer and closer to the golden ratio.

Q10 The sizes of the first 10 squares are 1, 1, 2, 3, 5, 8, 13, 21, 35, and 56. These are the Fibonacci numbers. (The ratio between successive Fibonacci numbers gets closer and closer to the golden ratio.)

WHOLE-CLASS PRESENTATION

Use the **Golden Rectangle Present.gsp** sketch to demonstrate golden rectangles and golden spirals.

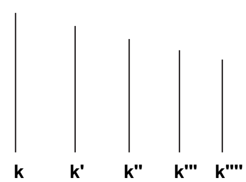
Use page 1 to show the steps of the construction through step 12. After clicking the buttons, drag point *C* to make the ratios equal. On page 2 you can use a tool to build the first golden spiral, covering steps 13 through 16. Page 3 is for the Explore More construction. Page 4 shows the construction used to create the custom tool.

Fractals and Ratios

A *fractal* is a geometric figure with limitless complexity. Whenever you look at some small detail, you find an even smaller detail. The fractals you will investigate here are also *self-similar.* This means that small parts of the figure are actually scale replicas of the entire figure.

RATIO OF SIMILARITY

First, you will look at the similarity of line segments. Two objects are *similar* if they have the same shape but not necessarily the same size. All line segments are similar because they all have the same shape. The *ratio of similarity* between two line segments is the ratio of their lengths.



1. Open **Fractals and Ratios.gsp.** The first page contains only line segment k and point C. Think of this as the first post in a fence that continues along a highway toward point C. If you take a photograph of the fence, the second post will appear smaller in the distance.

To mark the center, select the point and choose **Transform | Mark Center.**

2. Mark point C as the center for dilation.
3. Dilate segment k by selecting it and choosing **Transform | Dilate.** In the dialog box that appears, define the scale factor as the ratio 9/10. A new, smaller segment appears, representing the next fence post.

To show the labels, select the segments and choose **Display | Show Labels.**

4. Leaving the new line segment selected, dilate it by the same ratio. Continue dilating until there are at least four line segments on the screen. Show the labels of all the segments.
5. Drag k left and right and observe the effect on the dilated images.

The length of line segment k' is 9/10 the length of line segment k. This is the ratio of similarity of $k' : k$. You can confirm this ratio by measuring it.

To measure the ratio, select in order k' and k. Then choose **Measure | Ratio.**

6. Measure the ratio of k' to k.

Q1 What are the following ratios of similarity? Calculate them on paper. Then check your answers by choosing **Measure | Ratio.**

a. $k'':k'$ b. $k'':k$

c. $k''':k$ d. $k''':k'$

Q2 Drag k again. Do the measurements change?

ITERATION

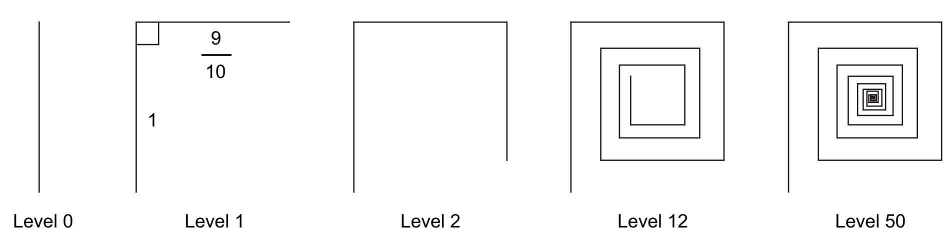
Next you will use iterated constructions to draw some fractals. In the previous construction, you repeated a simple dilation several times. Each repetition is an *iteration.* Now you will have the computer do the iterations for you.

7. Go to the Fence Post 2 page. In this sketch, the endpoints of the line segment have been dilated once. You will use the parameter *depth* to control the number of iterations.
8. Select in order point A, point B, and parameter *depth.* Hold down the Shift key and choose **Transform | Iterate To Depth.**
9. A dialog box appears asking how to map the two points. Answer by clicking the correct points in the sketch. Map point A to D, and map B to E. From the Structure pop-up menu, choose **Only Non-Point Images.** Click Iterate.
10. To increase the depth of the iteration, select the parameter *depth* and press the + key.

You can use Sketchpad's **Dilation Arrow** tool to zoom in on point C. Double-click C to mark it as the center, and then drag the fence posts away from C.

Q3 As you increase the depth, each new level of the fractal is closer to point C. How many iterations does it take for the fractal to reach point C? What would you see if you "zoomed in" and got a closer look at the fence closer to point C?

The figure below shows the first few levels of a different fractal. It begins as a single line segment. This line segment is rotated 90° about one endpoint, then dilated by a ratio of 9/10. That brings it to level 1. To reach level 2, the same thing is done to the new segment. Each new iteration adds one line segment.



Q4 How is this fractal different from the fence post fractal? How is it the same?

11. Open the Rectangular Spiral page. You will see the original segment and the endpoints of the second one.
12. Select in order point *A*, point *B*, and parameter *depth.* Hold down the Shift key and choose **Transform | Iterate To Depth.** This time map *A* to *B* and *B* to *C.* As before, choose **Only Non-Point Images.** Click Iterate. Increase the depth to at least 20.

SELF-SIMILARITY

13. Press the *Show Similar Fractal* button.

Q5 Press the red *Level* buttons to align segment *DE* with different levels of the fractal that you constructed. Describe what you see. How does this relate to self-similarity?

Q6 Measure the ratio *DE*/*AB*. Use the red buttons again and record the ratio of similarity for levels 0 through 4 of your fractal.

Q7 The construction is based on the ratio 9/10. Change it to 7/10. Now what is the ratio of similarity for levels 0 through 4? Use the red buttons to check your answers.

Objective: Students use iterated constructions to build fractals. They observe the property of self-similarity and measure the ratio of similarity.

Student Audience: Algebra 1/Algebra 2

Prerequisites: A conceptual understanding of geometric similarity is important, but an ability to prove similarity is not. You can use this activity as an introduction to iterative algorithms.

Sketchpad Level: Intermediate. There are two iterated constructions to perform.

Activity Time: 30 minutes

Setting: Paired/Individual Activity (use **Fractals and Ratios.gsp**) or Whole-Class Presentation (use **Fractals and Ratios Present.gsp**)

The image beside the introductory paragraph is Barnsley's Fern. You can research it on the web. (The activity book *Exploring Precalculus with The Geometer's Sketchpad* contains a supplemental activity, on the CD only, that allows students to construct their own fractal fern.)

RATIO OF SIMILARITY

The description of the term *similarity* here is rather vague. A more precise definition would mention the requirement that angles must be equal and corresponding distances must be in proportion. Consider addressing the meaning of this term in a class discussion.

Q1 Using fraction notation here would help to reinforce the concept of the ratio, but all of the ratio measurements will be in decimals.

a. 9/10 = 0.9 b. 81/100 = 0.81

c. 729/1000 = 0.729 d. 81/100 = 0.81

Q2 When you drag *k,* the positions of the posts change, but their heights and ratios remain constant.

ITERATION

Q3 Each iteration moves only one-tenth of the remaining distance to the point, so no finite number of iterations will bring it to the point. (Because of the limited resolution of computer screens, students may think that the fractal does reach point *C* after a large number of iterations. Try to elicit from students convincing arguments that the fractal can never reach point *C*.) This is a good illustration of the concept of limitless complexity. Because of this limitless complexity, you cannot construct an entire fractal. You can only define rules for the drawing and carry them out until the detail gets too small to see.

If you could zoom in on point *C,* you would see the same image. Using the fence post analogy again, this is like driving another mile down the road. The row of posts will still look the same. Point *C* is on the horizon, and you can never reach the horizon.

Q4 The rectangular spiral is a rearrangement of the parts of the fence post fractal. The lengths of the new segments are the same as those in the fence post fractal, but each is perpendicular to the previous segment rather than parallel to it.

SELF-SIMILARITY

Q5 When *DE* is matched to one of the line segments of the black fractal, the red fractal matches all of the higher levels. This demonstrates that small parts of the black fractal are similar to the whole.

Q6 1.0 (level 0) 9/10 = 0.9 (level 1)

81/100 = 0.81 (level 2)

729/1000 = 0.729 (level 3)

6561/10000 = 0.6561 (level 4)

Q7 1.0 (level 0) 7/10 = 0.7 (level 1)

49/100 = 0.49 (level 2)

343/1000 = 0.343 (level 3)

2401/10000 = 0.2401 (level 4)

WHOLE-CLASS PRESENTATION

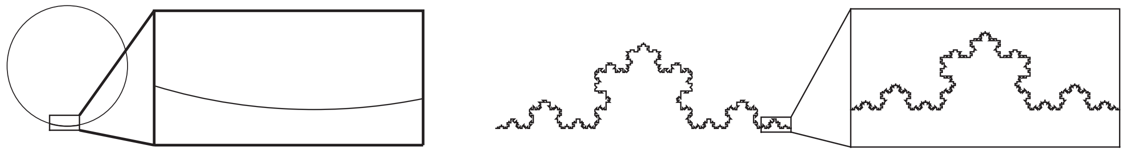
Open **Fractals and Ratios Present.gsp.** Press the button to show the second fence post, and then show the ratio of their heights ($k' : k$). Use the buttons to show more fence posts. Have students predict the ratios (from Q1 of the student activity), and then use the buttons to show them.

Similarly, use the provided buttons to show the iterated fence posts on page 2.

Length of the Koch Curve

When a fraction of a shape is similar to the entire object, the shape is called a *fractal.*

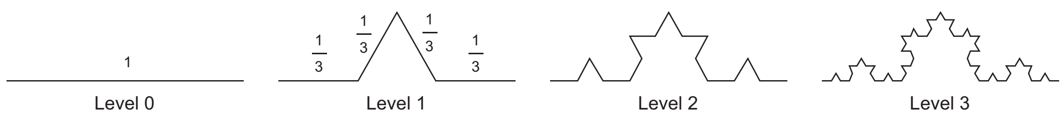
If you magnify a small fraction of an ordinary curve such as a circle, the small fraction appears straighter than the curve as a whole. This isn't true of fractals. For example, the Koch curve is always similarly bumpy, no matter how much you magnify it. In this activity you will create a Koch curve and investigate its length.



WHAT IS THE KOCH CURVE?

The Koch curve was first created by Swedish mathematician Helge von Koch (1870–1924).

The easiest way to describe a Koch curve is by using a *recursive* rule—a rule that is applied over and over again. Start with a segment (level 0) and divide it into thirds. Remove the middle third and replace it with two new segments, each equal in length to the removed segment (level 1). Apply this rule again to each new segment to see the next level of the Koch curve.



1. Open **Koch Curve.gsp.** This sketch already has the level 0 curve and the points you need to make the level 1 curve.

To hide the long segment, select it and choose **Display | Hide Segment.**

2. Hide the level 0 curve (the long segment).
3. Construct the four segments of the level 1 curve by using the **Segment** tool.

Q1 If the original level 0 segment is 1 unit long, how long is each level 1 segment?

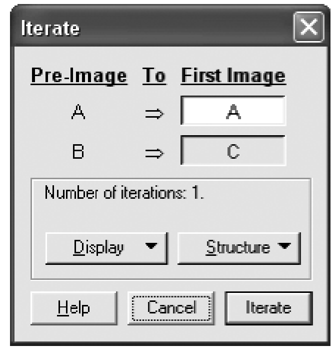
Q2 How long is the entire level 1 curve?

Creating many levels by hand would be time-consuming and error-prone. Instead, you will use Sketchpad's iteration feature to produce the levels automatically.

This page gives you a fresh level 0 starting place.

4. Go to the By Iteration page of the document.
5. Select in order point *A*, point *B*, and parameter *depth.* Hold down the Shift key and choose **Transform | Iterate To Depth.**

6. In the dialog box that appears, specify the endpoints of the first segment on which to iterate the construction. Start with the segment on the left, by clicking in the sketch on points *A* and *C*. The points appear in the First Image column.



7. To apply the same construction to points *C* and *D*, you must add a new map. Click the Structure button in the dialog box. In the pop-up menu that appears, choose **Add New Map.** Then click points *C* and *D* in the sketch.

8. Continue adding maps and mapping the points until you have done the construction on all four segments.

Pre-Image	To	Map #4	Map #3	Map #2	Map #1
A	⇒	*E*	*D*	*C*	*A*
B	⇒	*B*	*E*	*D*	*C*

To change the parameter, select it and press the **+** or **–** key. Don't go past 6 or your computer will slow down.

9. From the Display pop-up menu, choose **Final Iteration Only.** Then click Iterate.

10. Press the *Hide Level 0* button to hide the original segment and construction points. Change the depth parameter to see a higher level of the curve.

Q3 How long is each segment of the level 2 curve? How long is this entire curve?

Q4 Does this curve get longer at each level? If you keep applying the rule, how long will it eventually become? Will it run off the page?

Next you will compare the levels of the Koch curve.

Q5 On paper, make a table comparing levels of the curve from zero through four. How many line segments are there on each level? How long is each of them? What is the total length of the curve? Include this information in your table.

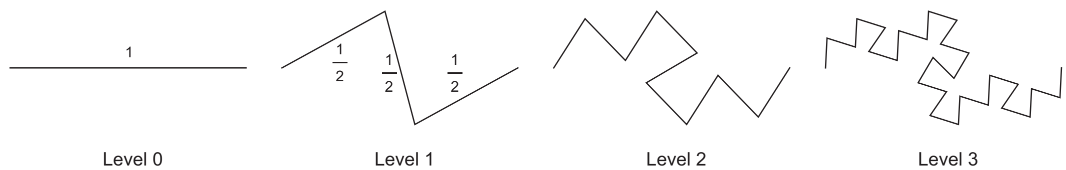
You can answer this more easily using exponents.

Q6 What is the total length of this curve at level 10?

Q7 Is there any limit to the length of this fractal?

EXPLORE MORE

Q8 On the Meander page is the setup for the meandering fractal below. This fractal also starts with a segment that is 1 unit in length. For each level after level 0, you replace each segment with three segments that are half as long. This iteration requires three mappings. Do the construction and answer the same questions (Q1–Q7) about its length.



Length of the Koch Curve

Objective: Students build a Koch curve and investigate its properties. They use fractions and exponents to calculate its length at various depths, and they make conjectures about its overall length.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students will find this easier if they have seen recursive rules in other contexts.

Sketchpad Level: Intermediate. This activity uses an iterated construction.

Activity Time: 20–30 minutes. Related projects are optional.

Setting: Paired/Individual Activity (use **Koch Curve.gsp**) or Whole-Class Presentation (use **Koch Curve Present.gsp**)

This activity reinforces and reviews various operations involving fractions and exponents while giving students a chance to construct an interesting and beautiful fractal at the same time. You can use the activity during a review of fractions and exponents, as part of a unit on fractals, or as an engaging enrichment activity.

WHAT IS THE KOCH CURVE?

Q1 Each level 1 segment is 1/3 unit long.

Q2 The entire level 1 curve is 4/3 units long.

Q3 Each level 2 segment is 1/9 unit long. There are 16 such segments, so the entire curve is 16/9 units long.

Q4 At this point students are making conjectures. Try to use the conjectures to get students to discuss and explain their reasoning, without emphasizing the "right" answers. (In fact, it does get longer at each level, and there is no limit to its length. There are limits to its range: It would never outgrow the page.)

Q5

Level	# Segments	Segment Length	Total Length
0	1	1	1
1	4	1/3	4/3
2	16	1/9	16/9
3	64	1/27	64/27
4	256	1/81	256/81

Q6 The curve length at level 10 is $(4/3)^{10}$, which is about 18. The general formula for the length at level n is $(4/3)^n$.

Q7 There is no limit to the length of this fractal, which is interesting because the idea of a curve of unbounded length within a bounded region is counterintuitive. (Not all fractals have unbounded length. See the fence post and rectangular spiral fractals in the activity Fractals and Ratios.)

EXPLORE MORE

Q8 The length of the meander also grows at a geometric progression, but with ratio 3/2. At level n, the length of the fractal is $(3/2)^n$. As with the Koch curve, its length is without limit, but its range is limited.

WHOLE-CLASS PRESENTATION

Use the presentation sketch **Koch Curve Present.gsp** to demonstrate the construction and characteristics of the Koch curve. Then develop the table for Q5 as a class activity.

RELATED PROJECTS

P1 Research other fractals. Include in your report other ways of describing fractals and pictures of interesting fractals that you find.

P2 Research the history of fractals. When were fractals first described, and by whom? How are they used in movies and other forms of computer-generated graphics?

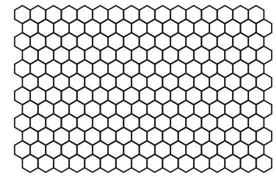
P3 Find a picture of the Sierpiński triangle, and try to figure out how to construct it using Sketchpad.

P4 Find other fractals that can be constructed with Sketchpad, construct them yourself, and present your constructions to your group or class.

P5 Research the term *fractal dimension,* and determine the fractal dimension of the Koch curve and of the meander.

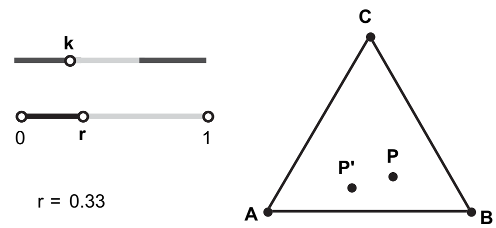
The Chaos Game

Patterns happen, often with no effort to make a particular pattern. If you mow a lawn, you will likely end up with a pattern in the grass. Bees have no notion of a hexagon but create a hexagonal tiling when they nestle their wax chambers as closely as possible. The Chaos Game is a study of some patterns that result from random choices.



THE SETUP

1. Open **Chaos Game.gsp.** Triangle ABC is an equilateral triangle. Point P is an independent point.



Q1 Drag point P across the screen. How is point P' related to P?

Q2 Point r controls ratio r. What changes occur when you drag point r? Where is P' when $r = 0$? Where is P' when $r = 1$? How is r related to the distance PP'?

Q3 Drag point k. What effect does that have on the sketch? Describe it in detail.

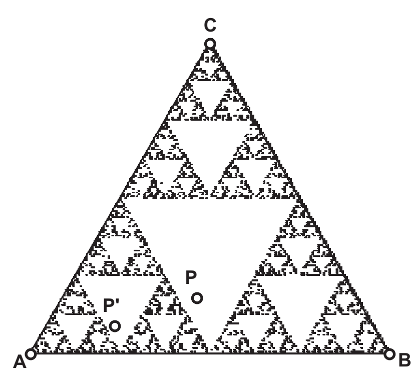
THE GAME

The idea of the Chaos Game is to start P moving and guess where it will go. First it goes to P'. From there, it will use the same ratio, r, and go toward another vertex. Which vertex? That's the random part. It could be any of the three.

2. Set the parameter *depth* to 1 and r to 0.50. Select in order points P and k and parameter *depth.* Hold down the Shift key and choose **Transform | Iterate To Depth.** A dialog box appears asking where the two points should be mapped. Answer by clicking the correct points. Map P to P', and map k to k. From the Structure pop-up menu, choose **To New Random Locations.** Click Iterate.

To animate the parameter *depth,* select it, choose **Display | Animate,** and use the Motion Controller to control the animation speed.

3. There is a new point on the screen. To see the next point, increase *depth* to two. Animate the *depth* parameter and observe the path as the point moves to new locations. This path is called the *orbit* of the point. Run the animation until *depth* is 1000 or more.

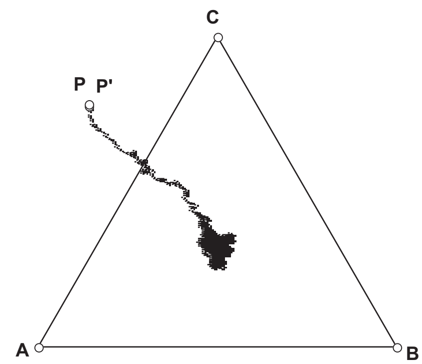


To change the parameter in one step, double-click on the parameter and enter the value.

4. Select the orbit. Choose **Display | Line Width | Dashed.** This will make the points smaller so that the pattern is easier to see.

Q4 Describe the pattern made by the orbit when $r = 0.50$. Drag point *P*. What effect does that have on the pattern?

You can still change many of the conditions after you have plotted the orbit. Drag point *B* to change the size of the triangle, drag *r* to change the ratio, and place *P* anywhere you wish. Following is a list of conditions that you can create. In each case, describe what you see and explain what causes the image to appear the way it does.



Q5 What does it look like when $r = 0$? What does it look like when $r = 1$? Explain what causes these orbit patterns.

To see a different random pattern, select the orbit and press the exclamation point (!) on your keyboard.

Q6 Start with point *P* inside the triangle and $r = 0.50$. There are many triangle patterns in the orbit. Slowly decrease *r*. What is *r* when the triangle patterns disappear?

In order to make fine adjustments to *r*, select point *r* and use the left and right arrow keys on the keyboard.

Q7 Start with point *P* inside the triangle. For what range of *r* does the orbit leave the triangle?

Q8 Drag point *B* so that the triangle is very small, only a few pixels across. Set *r* to 1.99, then 2.00, then 2.01. Describe the orbit.

EXPLORE MORE

So far, you have been working with an equilateral triangle. See what happens when you use some other triangle.

5. Select point *B*. Choose **Edit | Split Point From Ray.**
6. Select point *C*. Choose **Edit | Split Intersection From Circles.**

Q9 Now all of the vertices are independent points. Go back through the previous settings and see if your observations apply to other triangles.

Q10 In the same document, there are other pages with a square, a regular pentagon, and a regular hexagon. The rest of the setup is the same. Perform steps 2–4 again on these pages and see what patterns you can find.

Objective: Students create a recursive point mapping using a ratio and a random component. They then vary the ratio and observe changes in the patterns.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should understand the concept of the ratio of two distances.

Sketchpad Level: Intermediate. Students perform an iterated construction using objects in a prepared sketch.

Activity Time: 40 minutes+. This is the expected time required for the guided construction and questions. There are two extensions that would require more time.

Setting: Paired/Individual Activity (use **Chaos Game.gsp**) or Whole-Class Presentation (use **Chaos Game Present.gsp**)

Students should always be encouraged to explain their answers, but many of the things they observe in this activity will be difficult for them to fully grasp. You should expect them to write (or draw) a clear description of a pattern even if they are unable to explain why it appears.

THE SETUP

Q1 P' is in line with P and one of the triangle vertices.

Q2 Dragging point r changes the ratio r. When $r = 0$, P' is at P. When $r = 1$, P' is at one of the triangle vertices. The distance PP' is the distance from P to the vertex scaled by ratio r.

Q3 When you move point k, P and P' align with a different vertex of the triangle. The path of k is a line segment. The line segment is divided into thirds. Each third corresponds to one vertex of the triangle.

THE GAME

2. Demonstrate this step if possible.
3. This step calls for a depth of 1000, which might be a bit conservative. Some computers are fast enough to use much greater depth. (You can use **Edit | Advanced Preferences** to change the upper limit.) See what your computers can tolerate, and advise your students.

Q4 The pattern is similar to that of a Sierpiński triangle, which the students may have seen before.

In general, the position of point P appears to affect the first few points of the orbit, but there is very little difference in the overall pattern. In fact, changing P changes all the points, but the change is too small to notice in any but the first few iterations.

Q5 When $r = 0$, the pattern completely disappears. This is because P is mapped to itself on every iteration. The entire orbit is occupying only that one point.

When $r = 1$, the pattern disappears again, but for different reasons. Point P is mapped to a vertex. Throughout the orbit, it is jumping between the three vertices, but never anywhere else.

Q6 As r decreases, the triangle patterns overlap and the open regions are covered. When r reaches 0.33 (1/3), all of the open regions are covered and the triangle pattern is no longer visible.

Q7 When $r < 0$ or when $r > 1$, the orbit leaves the triangle. At values slightly higher than 1, there is a pattern with rotational symmetry.

When $r > 2$, the pattern changes. Point P essentially jumps over the vertices and ends up farther away and on the other side.

Q8 When $r = 1.99$, the orbit tends to stay near the triangle, and it forms a pattern with something close to 180° rotation symmetry.

When $r = 2.00$, the rotation symmetry is still there, but the orbit does not seem to be either attracted or repelled by the triangle. Also, close inspection will show that the points fall in a triangular grid, like isometric dot paper.

When $r = 2.01$, the pattern diverges in two different directions.

EXPLORE MORE

Q9 All of the observations from the previous sections hold true for any triangle.

Q10 The instructions are the same. Look for the Koch snowflake that appears on the Hexagon page.

WHOLE CLASS PRESENTATION

To present this activity to the entire class, follow the Presenter Notes and use the sketch **Chaos Game Present.gsp.**

In this presentation students observe a process in which a point is repeatedly dilated toward a randomly chosen point by a particular ratio. They see how such a random process can sometimes generate a regular pattern, depending on the ratio used.

1. Open **Chaos Game Present.gsp.**

Have several students answer each question in their own words.

Q1 Drag point P across the screen. Ask, "How is point P' related to P?"

Q2 Drag point r along its segment. Ask, "How does the value of r affect the relationship between P' and P?"

Q3 Drag point k. Ask "What effect does k have on the sketch?"

Q4 Ask, "What do you think would happen if we repeated this process starting with point P'?" Be sure to get several different predictions.

2. Press the *Show Iterated Image (constant k)* button to show the next image.

Q5 Ask, "What would happen if we did this over and over again?"

Use the **+** sign to increase *depth,* and the **–** sign to decrease it.

3. To test students' predictions, select parameter *depth* and press the + sign on the keyboard several times.
4. Hide the iterated image by pressing the *Hide Iterated Image (constant k)* button. Set *depth* back to 1 by pressing the – sign or by double-clicking the parameter and changing its value.

Q6 Ask, "What would happen if we did the same thing, but chose a random vertex at each step?" Get several different predictions concerning the pattern.

5. Press the *Show Iterated Image (random k)* button to show the next image. Increase the depth slowly, so students can observe how the next point goes halfway toward a randomly chosen vertex at each step.

Q7 Ask, "Can you see a pattern yet?" Students may or may not be able to make a prediction yet.

To stop the animation, press the *Animate Depth* button again.

6. Press the *Animate Depth* button, and let the animation run for a while. Stop when a pattern begins to emerge.

Q8 Have students report the pattern that they see. The greater the depth, the clearer the pattern will become. (Be careful increasing the depth; if you increase it too much, your computer will slow down.)

Set *depth* to a value at which your computer does not slow down too much.

Q9 Ask, "What do you think will happen if r is greater than 0.5? What if r is less than 0.5?" Drag r to investigate both situations. Ask students to explain the patterns they see based on the value of the ratio.

Use the remaining pages of the sketch to investigate patterns produced by similar iterations involving different numbers of vertices.

Exponents

In this activity you'll use a Sketchpad model to perform operations involving exponents.

MULTIPLICATION PATTERNS

1. Open **Exponents.gsp.**
2. Drag the red point (x) left and right, and observe the motion of the other points.

Q1 What do you think the points represent?

Q2 How many positions can you find that make all the points line up? What values of x do you think these positions indicate? Explain why the points should line up at these particular values.

Click the *Show Constants* and *Show Bars* buttons to show these objects.

3. Show the constants and the bars, and then drag point x again to observe the behavior of the bars and to check your results from Q1.

Are the differences in length constant from one bar to the next?

Q3 What pattern do the points make when $x > 1$? Why does this pattern make sense?

Q4 What pattern appears when $0 < x < 1$? Explain this pattern.

Q5 What pattern appears when $x < 0$? Explain this pattern.

EXPONENTS

With addition, when you have a problem like $x + x + x + x + x + x + x$, you can write the problem more easily using multiplication:

$$x + x + x + x + x + x + x = 7x$$

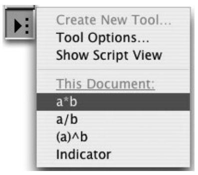
Similarly, when you have a multiplication problem like $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$, you can write the problem more easily using an exponent:

$$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$$

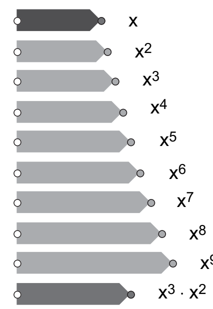
4. Go to the Multiplication page. This page uses exponents to write the multiplication problems more simply. Drag point x left and right to make sure you observe the same patterns as you did on page 1.

MULTIPLYING AND DIVIDING

In the next several steps, you'll investigate what happens when you multiply or divide two of the values represented by the bars.



5. Choose the **a*b** custom tool from the Custom Tools menu. This tool will allow you to multiply any of the exponent bars.



6. Use the **a*b** tool to multiply x^3 by x^2, by clicking on 5 objects in order: the first unused point below all the bars, the point at the tip of the x^3 bar, the x^3 label, the point at the tip of the x^2 bar, and the x^2 label.

Q6 Is the $x^3 \cdot x^2$ bar the same length as any of the existing bars? Test your answer. Click the **Indicator** custom tool on the tip of the $x^3 \cdot x^2$ bar. Then use the **Arrow** tool to drag x left and right. Describe your observations.

Q7 Create a bar for $x^4 \cdot x^3$, and determine what existing bar it matches.

When you test, be sure to drag x left and right to try different values.

Q8 Make a conjecture concerning the result of $x^a \cdot x^b$, and test your conjecture by constructing another multiplication problem.

7. Go to the Division page. Use this page to explore division problems.

To use this tool, click it on 5 objects in order, just as you did with the **a*b** tool.

Q9 Use the **a / b** tool to create a bar for $\frac{x^7}{x^3}$. What is the result?

Q10 Make a conjecture concerning the result of $\frac{x^a}{x^b}$, and test your conjecture by constructing another division problem.

RAISING TO A POWER

8. Go to the Power page. You will use this page to explore problems like $(x^4)^2$.

To use this tool, click the starting position, the point and label for x^4, and the point and label for the constant 2.

Q11 Use the **(a)^b** tool to create a bar for $(x^4)^2$. What is the result?

Q12 Make a conjecture concerning the result of $(x^a)^b$. Test your conjecture by constructing another similar problem.

Q13 Summarize your conjectures from Q8, Q10, and Q12. Include an example for each conjecture. Use the definition of exponents (as repeated multiplication) to explain why each of your conjectures makes sense.

EXPLORE MORE

Q14 What if the bases are not the same? Is there a rule you can use for problems like $x^a \cdot y^b$? Go to the Explore page and experiment. Describe your conclusions.

Objective: Students use a Sketchpad model to explore repeated multiplication, and the multiplication, division, and power properties of exponents.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: It's best if students already have some familiarity with using exponents.

Sketchpad Level: Intermediate. Students manipulate a pre-made sketch and use several custom tools.

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Exponents.gsp** in either setting)

MULTIPLICATION PATTERNS

Q1 Answers will vary. This question encourages students to observe closely and think about the patterns they see. The points represent various powers of x, with the vertical position corresponding to the power and the horizontal position corresponding to the value of x.

Q2 There are two positions of x at which all the points line up: $x = 1$ and $x = 0$. Repeatedly multiplying either 1 or 0 by itself continues to give the same result.

Q3 When $x > 1$, the points move increasingly rightward as you go down the screen, showing that the value of x^n increases more and more quickly for larger and larger values of n. Be sure students notice that the differences in the lengths of the bars are not constant, but increasing.

Q4 When $0 < x < 1$, the points move toward the left as they go down, approaching a straight line (vertical asymptote) at $x = 0$. This pattern makes sense because multiplying by a value less than one always gives a result that is closer to zero than the number you started with. Be sure students notice that the differences in the lengths of the bars are not constant, but decreasing.

Q5 When $x < 0$, the bars alternate between the left and right sides, because multiplying a number by a negative value always gives a result with a sign opposite to the sign of the original number.

MULTIPLYING AND DIVIDING

Q6 The $x^3 \cdot x^2$ bar is the same length as the x^5 bar no matter how you drag the value of x. This makes sense, because you've multiplied three x's by two more x's, so that there are now five x's multiplied together.

Q7 The bar for $x^4 \cdot x^3$ is the same length as the x^7 bar.

Q8 Conjecture: The bar for $x^a \cdot x^b$ is the same length as the x^{a+b} bar. Students will construct different problems to test this conjecture.

Q9 When you use the **a / b** tool to create a bar for x^7 / x^3, the resulting bar is the same length as the x^4 bar.

Q10 Conjecture: The bar for x^a / x^b is the same length as the x^{a-b} bar. Students will construct different problems to test this conjecture.

RAISING TO A POWER

Q11 When you use the **(a)^b** tool to create a bar for $(x^4)^2$, the result matches x^8.

Q12 Conjecture: The bar for $(x^a)^b$ is the same length as an x^{ab} bar. Students will construct different problems to test this conjecture.

Q13 $x^a \cdot x^b = x^{a+b}$: Because the first factor (x^a) represents a values of x multiplied together, and the second (x^b) represents b values of x multiplied, the product has $a + b$ values all multiplied together.

$x^a / x^b = x^{a-b}$: When you divide, the first b factors of x in the numerator will cancel out with the factors in the denominator, leaving the $a - b$ factors of x as the result.

$(x^a)^b = x^{ab}$: The second exponent (b) means that there are b factors to multiply together, where each factor is x^a. The total number of factors of x is ab, so this is the exponent in the result.

EXPLORE MORE

Q14 If the bases are not the same, there is no rule you can use to simplify a problem like $x^a \cdot y^b$. Although there are some special cases that may suggest a more general result, testing with many values of x, y, a, and b will show that there is no general pattern.

In this presentation students use a visual representation of powers of x to understand and explain the exponent rules that apply to the expressions $x^a \cdot x^b$, x^a / x^b, and $(x^a)^b$.

1. Open **Exponents.gsp.** Drag the red point (x) left and right.

Q1 Ask, “What do you observe? What do you think the points represent?”

Q2 Continue dragging slowly, and ask, “How many positions are there that make all the points line up? What values of x do you think these positions indicate?” Have students explain why the points should line up at these particular values.

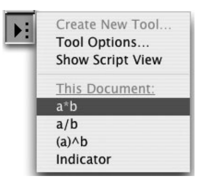
2. Click the *Show Constants* button and the *Show Bars* button, and drag x again.

Are the differences in length constant from one bar to the next?

Q3 Ask students to describe and explain the patterns they observe when $x > 1$, when $0 < x < 1$, and when $x < 0$.

3. Remind students that the shortcut for an expression like $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ is x^7. Then go to the Multiplication page of the sketch. Drag point x left and right to make sure students observe the same patterns as they did on page 1.

In the next several steps, you’ll demonstrate what happens when you multiply or divide two of the values represented by the bars.



4. Choose the **a*b** custom tool from the Custom Tools menu. Explain that this tool allows you to multiply any of the exponent bars.

5. Use the **a*b** tool to multiply x^3 by x^2, by clicking on 5 objects in order: the first unused point below all the bars, the point at the tip of the x^3 bar, the x^3 caption, the point at the tip of the x^2 bar, and the x^2 caption.

Q6 Ask, “Is the $x^3 \cdot x^2$ bar the same length as any of the existing bars?” Drag x back and forth to give students a chance to observe. (They should answer x^5.)

6. To confirm this answer, choose the **Indicator** custom tool and click it on the tip of the $x^3 \cdot x^2$ bar. Then use the **Arrow** tool to drag x left and right.

7. Create a bar for $x^4 \cdot x^3$, and ask students what existing bar it matches. (x^7)

Q7 Have students write out both x^4 and x^3 using repeated multiplication, and ask them how this way of writing it explains the way the bars behave.

Q8 Ask students to make a conjecture concerning the result of $x^a \cdot x^b$, and to explain why the conjecture makes sense.

Use the Division and Power pages to investigate x^a / x^b and $(x^a)^b$.

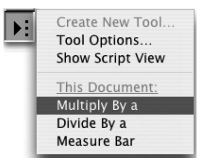
Zero and Negative Exponents

By now you should be comfortable doing calculations with exponents that are positive integers. From here, certain questions naturally arise. What if the exponent is zero? What if it is negative? What if it is not an integer? This activity explores the concept of zero and negative exponents. Non-integer exponents will have to wait.

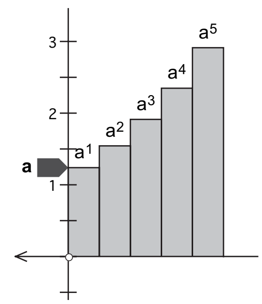
POSITIVE EXPONENTS

1. Open **Zero Exponents.gsp.**

The bar represents the number a. You can drag the marker to change its value. The label on the bar is a^1, which is the same thing as a.



2. Start with a between 1 and 1.5. You can change it later. Now multiply a^1 by a. Press and hold the **Custom** tools icon to display the Custom Tools menu. Choose the **Multiply By *a*** tool. This tool works by itself, so there is no need to click on anything.



Another vertical bar appears representing a^2. You get this result because $a^1 \cdot a = a^2$.

Q1 What is the result if you multiply a^2 by a?

3. Choose **Multiply By *a*** again. Use the tool several times and study the result.

Q2 Consider the number a^n, where n is a positive integer. What happens when you multiply the number by a? State a general rule. Explain why this is true.

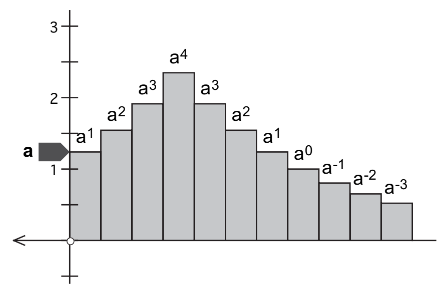
Q3 As the exponents increase, do the heights of the bars increase by the same amount each time? How can you tell? Explain your observations.

ZERO EXPONENT

4. Go to page 2. This is the same sketch. The progression of bars goes up to a^4.

5. Choose the custom tool **Divide By *a*.** It does just what the name says.

Q4 What is $a^4 \div a$? What happens when you divide a^n by a? Explain why this is true.



6. Use the **Divide By *a*** tool two more times, so that the progression runs down to a^1. Dividing by a once more should give you a^0. Try it.

Q5 What is the value of a^0? Drag the marker to test different values of a.

NEGATIVE EXPONENTS

Q6 At this point, a^0 should be the last number in the progression you are building. Using your answer to Q3, what will be the result when you divide by a again? Choose **Divide By *a*** and check your answer.

Q7 Starting with the number 1, if you divide by a three times, that is the same as dividing by a^3.

$$1 \div a \div a \div a = 1 \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a^3}$$

How can you write this same number with a negative exponent? Use the sketch to check your answer.

EXPLORE DIFFERENT BASES

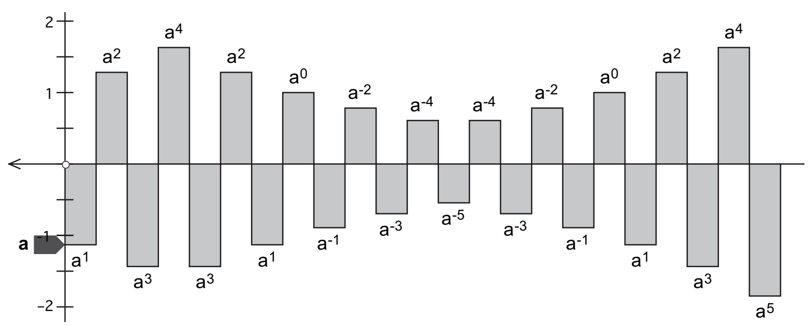
The point of this activity was to investigate zero and negative exponents, but you may have noticed some interesting changes that occur when you change the base, a. Drag the marker to change the value of a, and answer the following questions.

Q8 Start with $a > 1$. If you keep multiplying the bar lengths by a, is there a limit to how high the bars will go? If you keep dividing them by a, is there a limit to how short the bars will become?

Q9 Pull the marker downward so that $0 < a < 1$. What change do you see in the pattern formed by the bars? Explain why this is.

Q10 As you change a, which bars do not change at all? Why?

Q11 When a is negative, you will see an entirely different pattern. Describe the pattern, and explain why it looks this way.



Q12 Play the game on the Simplify page at least four times. Each time, write down the original problem and the solution. Try writing the solution down before you actually drag the variables.

Objective: Students create a sequence of bars to compare various integer powers of a given base. From the pattern formed, they learn to interpret zero and negative exponents.

Student Audience: Algebra 1

Prerequisites: Students must understand the concept of raising a number to a positive integer power. Knowledge of zero and negative exponents is helpful, but not necessary.

Sketchpad Level: Easy. Students use prepared custom tools.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Zero Exponents.gsp**) or Whole-Class Presentation (use **Zero Exponents Present.gsp**)

POSITIVE EXPONENTS

2. The tools will work for any nonzero setting of a, but the activity creates an exponential sequence, so the heights of the rectangles can get out of hand if a is much more than 1.5.

Q1 $a^2 \cdot a = a^3$

Q2 Multiplying by a raises the power by one: $a^n \cdot a = a^{n+1}$. This is true because $a^n = a \cdot a \cdot a \cdots a$, where a appears as a factor n times. If you multiply this by one more a, there will be $n + 1$ factors.

Q3 As the exponents increase, the heights of the bars do not increase by the same amount each time. Provided $a > 1$, they increase by more and more each time. This is more obvious if you change the scale by dragging one of the tick numbers on the axis. Students may use specific examples to explain: "Multiplying 1 by 1.37 adds only 0.37 to the bar height, but multiplying 10 by 1.37 results in 13.7, adding 3.7 to the original height."

ZERO EXPONENT

Q4 $a^4 \div a = a^3$, and generally, $a^n \div a = a^{n-1}$. As in Q2, $a^n = a \cdot a \cdot a \cdots a$, where a appears as a factor n times. If you divide this by a, you cancel the last factor, leaving $(n - 1)$ factors.

Q5 $a^0 = 1$. This follows from the fact that $a^1 \div a = 1$.

NEGATIVE EXPONENTS

Q6 $a^0 \div a = a^{-1}$, using the rule from Q3.

Q7 Since $1 = a^0$, dividing by a three times is equivalent to dropping the exponent three times to a^{-3}. Therefore, $1/a^3 = a^{-3}$.

EXPLORE DIFFERENT BASES

In this extension students can use the same sketch to see the effects of changing the base.

Q8 If $a > 1$, higher exponents always correspond to larger rectangles, hence, larger numbers. As you keep multiplying, there is no limit to how high the bars will go. Similarly, if you divide repeatedly, there is no limit to how short the bars will become.

Q9 If $0 < a < 1$, higher exponents correspond to smaller numbers. This follows from the fact that multiplying any positive number by a number greater than one increases it, while multiplying it by a number between one and zero makes it smaller. In either case, the result is always positive.

Q10 As a changes, any bar with the value a^0 remains constant. This is because $a^0 = 1$.

Q11 When a is less than zero, the bars alternate between positive and negative. The numbers with odd exponents are negative. Those with even exponents are positive.

In each step of the activity, students formed the next number by either multiplying or dividing by a negative number, a. Multiplying or dividing by a negative changes the sign, thus creating the alternating pattern.

Q12 Problems and solutions to the Simplify game vary.

(Both the activity document and the presentation document have another custom tool, **Measure Bar,** which was not used in the activity. Choose the tool and click on one of the bars. It will display the number represented by the bar.)

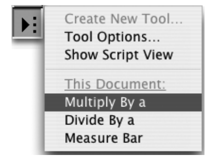
Zero and Negative Exponents

Presenter Notes

In this presentation students observe the visual pattern formed when an exponent increases as a number is repeatedly raised to higher powers, and observe the related pattern as the exponent is reduced first to zero and then to negative values.

Explain that the label on the bar is a^1, which is the same thing as *a*.

1. Open **Zero Exponents Present.gsp.** Drag the marker so students can see how it changes the value of *a*. Return the marker to its original position.
2. To multiply the value represented by the first bar by *a*, press and hold the **Custom** tools icon to display the Custom Tools menu. Choose the **Multiply By *a*** tool. This tool works by itself, so there is no need to click on anything.



Q1 Ask students what the new bar represents. Drag the marker to change the value of *a* to 2, so that students can see the new bar has the value 4. Return the marker to its original position before continuing.

Q2 Ask, “What will be the result if we multiply a^2 by *a*?”

3. Choose **Multiply By *a*** again. Use the tool several times.

Q3 Ask, “As the exponents increase, do the heights of the bars increase by the same amount each time? How can you tell?”

4. Choose the custom tool **Divide By *a*.**

Q4 Ask students to explain why the new bar is the height that it is.

5. Use **Divide By *a*** several more times, until the progression runs down to a^0.

Q5 Ask, “What do you think is the value of a^0?” Drag the marker so that students can see that the value of *a* has no effect on this bar.

Q6 Ask, “What will happen if we divide by *a* again?”

6. Choose **Divide By *a*.**

Q7 Discuss what the resulting bar represents. Choose **Divide By *a*** twice more during the discussion. Try to get students to propose formulations like these:

$$a^{-1} = 1 \div a = \frac{1}{a} \quad \text{and} \quad a^{-3} = 1 \div a \div a \div a = 1 \cdot \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a^3}$$

Use the other numbered pages to create different patterns that involve both positive and negative exponents.

Use the Simplify page to give students practice in manipulating expressions to eliminate negative exponents.