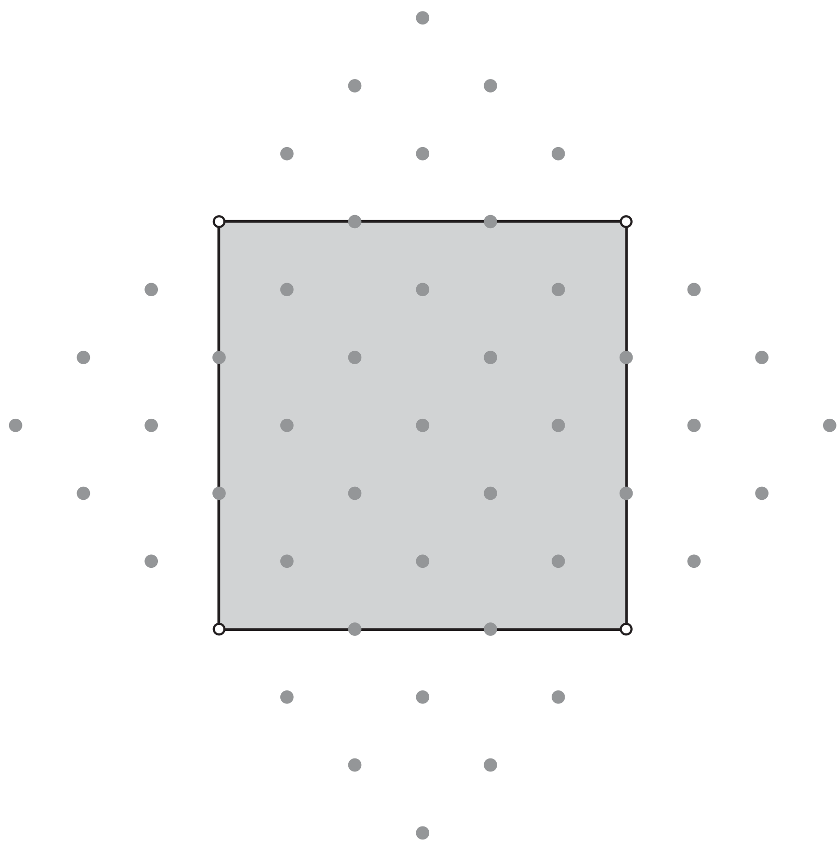
3

Algebraic Expressions



Order of Operations

In this activity you will explore the importance of agreeing on and following the standard rules for order of operations:

1. Evaluate parentheses.
2. Evaluate any exponents.
3. Perform multiplication and division from left to right.
4. Perform addition and subtraction from left to right.

Image courtesy of U.S. Postal Service

DIFFERENT STROKES FOR DIFFERENT FOLKS

All known cultures in the galaxy evaluate parentheses and exponents before performing addition, subtraction, multiplication, and division.

When you perform a series of arithmetic operations on Earth, you follow the Terran order described above. This is not the only order possible. The rules for steps 3 and 4 are different in Vulcan mathematics, and different still in Klingon mathematics.

Terran	Vulcan	Klingon
3. Multiply and divide from left to right. 4. Add and subtract from left to right.	3. Add and subtract from left to right. 4. Multiply and divide from left to right.	3. Add, subtract, multiply, and divide from left to right, in the order in which they appear.

For this reason, you, Spock (who is a Vulcan), and Worf (who is a Klingon) may all get different answers to the same problem.

1. Open **Order of Operations.gsp.**
2. On the first page, you will evaluate the expression $1 + 3 \cdot 2$ in all three systems. Think of this as $a + b \cdot c$. To prepare, drag point a to 1, b to 3, and c to 2.

On Earth, multiplication and division must be done first, from left to right, before addition and subtraction. You'll do this calculation on the Terran number line.

When you use the **Multiply** tool, click the point on the number line, not the marker.

With the marker selected, choose **Display | Color.**

3. On the Terran number line, you must calculate $b \cdot c$ first. Press and hold the **Custom** tools icon, choose the **Multiply** tool, and click on points b and c.
4. Now calculate $a + (b \cdot c)$ on the Terran number line by choosing the **Add** custom tool and clicking first on point a and then on point $(b \cdot c)$. When you finish, change the color of the resulting marker.
5. In the Vulcan system, they do addition and subtraction first, before multiplication and division. Use the **Add** and **Multiply** custom tools to do the Vulcan calculation on the second number line. Color the final result marker to match the color of the Terran answer.

6. Similarly, do the Klingon calculation and color its final result marker.

Q1 Which of the three systems get the same result for this calculation?

Q2 Drag points *a, b,* and *c*. Are there any values of *a, b,* and *c* for which all three calculations are equal? If so, what are they?

7. Go to page 2 and evaluate $12 \div 2.5 - 1$ in all three systems. Prepare by dragging point *a* to 12, *b* to 2.5, and *c* to 1. Then use the custom tools.

Q3 What answer do the various systems give for this calculation? Which ones agree?

Q4 If some systems agree, are they the same ones that agreed for the first expression you evaluated?

Q5 Are there some values of *a, b,* and *c* for which all three calculations are equal?

Q6 Try to construct a mathematical expression for which Terrans and Vulcans agree, but Klingons get a different result. What did you find?

Q7 None of the three cultures is willing to abandon its system to adopt one of the others. In this situation, how do you think mathematicians from the three different cultures could communicate? Is it possible to create a Galactic Standard that beings from all three cultures would understand the same way?

Q8 Explain in your own words why we need rules to specify the order of arithmetic operations. What problems could arise if different people used different rules?

EXPLORE MORE

For these four questions, duplicate the Explore More page four times by choosing **File | Document Options** and using the Add Page button.

Even though the rules are different, sometimes Terrans get the same results as either Vulcans or Klingons. Perform each calculation below in all three systems. List the Terran result in the first blank column below. In the next column, list any other systems that agree with the Terran result, and in the last column, list systems that disagree.

	Calculation	Terran Result	Systems Agreeing	Systems Disagreeing
Q9	$6 - 4 \div 2$			
Q10	$7 + 5 + 2$			
Q11	$2 \cdot 4 - 3$			
Q12	$14 \div 7 \cdot 3$			

Q13 Research the RPN system and evaluate it as a possible Galactic Standard.

Q14 On page abcd, create a problem for which each system gets a different answer.

Objective: Students consider the possibility that different galactic cultures have different rules for evaluating expressions. The resulting discrepancies emphasize that mathematical communication depends on people agreeing on and abiding by standard rules for the order of operations.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students need basic arithmetic skills and an introduction to the order of operations.

Sketchpad Level: Intermediate. Students duplicate pages and use prepared custom tools.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Order of Operations.gsp**) or Whole-Class Presentation (use **Order of Operations Present.gsp**)

DIFFERENT STROKES FOR DIFFERENT FOLKS

Q1 The systems differ. The Terran result is 7, but both Vulcans and Klingons get a result of 8.

Q2 All three cultures get the same result if $a = 0$ or if $c = 1$.

Q3 For this problem, Terrans and Klingons get a result of 3.8, and Vulcans get 8.

Q4 Vulcans and Klingons agree on Q1, but Terrans and Klingons agree on Q2. Different cultures agree, depending on the problem.

Q5 All three calculations are the same when $c = 0$. Many other combinations of values exist.

Q6 Answers will vary, but all answers require four values. One example is $3 + 1 \cdot (-1) + 2$. The result is 4 for both Terrans and Vulcans, but not for Klingons.

Q7 Answers will vary. This question is a good jumping-off place for a class discussion. Students may notice that agreement on rule 1 (evaluate parentheses first) means that mathematicians from all three cultures can eliminate ambiguity by inserting enough parentheses to completely determine the order of operations.

Q8 Answers will vary. A lack of rules could result in many disagreements in retail transactions, in business deals, and throughout the economy, and could handicap scientists, engineers, and anyone who calculates frequently.

EXPLORE MORE

Q9 $6 - 4 \div 2 = 4$. No other culture agrees with this Terran result; both Klingons and Vulcans get 1.

Q10 $7 + 5 + 2 = 14$. All three cultures agree on this result.

Q11 $2 \cdot 4 - 3 = 5$. Klingons and Terrans agree on the result, but Vulcans get an answer of 2.

Q12 $14 \div 7 \cdot 3 = 6$. All three cultures agree on this result.

Q13 RPN is an unambiguous algebraic notation that does not use parentheses. Based on work by Polish mathematician Jan Lukasiewicz in the 1920s, it is supported by quite a few calculators, particularly from HP. Much more information is available on the web.

Q14 On the earlier pages you performed only two operations, so there were only two possible answers. Page abcd has four variables, making it convenient to create an expression involving three operations. It's not hard to create such an expression for which all three systems get different answers. An example is $1 + 2 \cdot 3 - 4$, which comes out to 3 in Terran, -3 in Vulcan, and 5 in Klingon.

WHOLE-CLASS PRESENTATION

In this presentation students will see the results of applying different rules for the order of operations, and will gain a greater appreciation for the importance of agreeing on standard rules.

Use the Presenter Notes, along with the sketch **Order of Operations Present.gsp,** to present this activity to the whole class. As you proceed with the presentation, you will need to refer to the rules that are used in the Terran, Vulcan, and Klingon civilizations. You can see the rules at any time by pressing the *Show Rules* button, and you can return to the presentation by pressing the *Return to Presentation* button.

You can use the custom tools to create additional examples. To get different results for all three cultures, you need to use a problem involving at least three operations. You can use page abcd to create such a problem. See Q14 for an example.

In this presentation you will explore the different rules for order of operations that are observed in the Terran, Vulcan, and Klingon galactic civilizations.

Ask the class if it really makes a difference. Although the three groups use a different order, they are still performing the same operations. Won't the answers be the same? Put this question to the test by evaluating the expression $1 + 3 \cdot 2$, but using variables so we can change the numbers later.

1. Open **Order of Operations Present.gsp.** Drag the *a, b,* and *c* markers so that the class can see how the corresponding markers move in unison. Set up the problem by pressing the *Move Points* button.

Press the *Show Rules* button to display the rules for each system.

Q1 Begin with the Terran system. Ask, "Which operation comes first?" In the Terran system, it is multiplication. Press the *Terran Step 1* button to show this step.

Q2 Ask, "Which operation is next?" Press *Terran Step 2* to show the addition.

Q3 Ask, "Will Vulcans get the same answer? Which operation comes first?" By Vulcan rules, the addition comes first, and multiplication second.

2. Press *Vulcan Step 1* to show $a + b$. Then press *Vulcan Step 2* to show $(a + b) \cdot c$.

Q4 Ask, "What order will Klingons use?" (Like Vulcans, Klingons will add first and then multiply.)

3. Press *Klingon Step 1* and *Klingon Step 2* to show the Klingon result.

Since Vulcans and Klingons used the same order, their answers for this problem will match for any numbers.

Q5 Ask, "If we start with different numbers, will Terrans always differ from the others? Will Vulcans and Klingons always agree?" Drag *a, b,* and *c* to test student responses. The Terran answer agrees with the others if $a = 0$ or $c = 1$.

Q6 Ask, "Does this mean Vulcans and Klingons always agree, and Terrans always disagree?"

4. Go to page 2 and evaluate $12 \div 2.5 - 1$ in all three systems. This time Terrans and Klingons agree on $((a \div b) - c) = 3.8$, but Vulcans get $(a \div (b - c)) = 8$.

Q7 Again ask for values that make all three systems agree on this calculation. One solution is $c = 0$.

Discuss the reasons for having rules for the order of operations. A good comparison is the rules for driving on the right or left side of a road. Both systems are used effectively on this planet. The important thing is that everyone in a given society observes the same rule.

The Microsoft Calculator uses Klingon when the view is set to Standard, but uses Terran when it's set to Scientific.

Ask the students if they know of any place on Earth where the Klingon rules prevail. Many inexpensive pocket calculators take operations in order, with no hierarchy. So does the Microsoft Calculator program, which is installed with Windows.

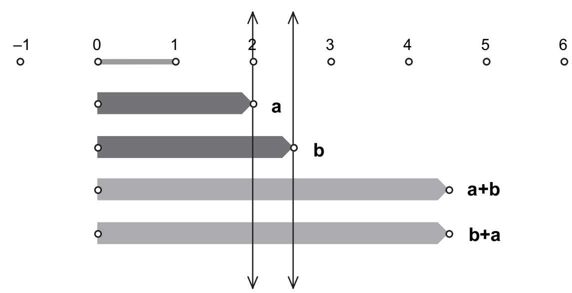
Equivalent Expressions

In this activity you will investigate algebraic expressions that are equivalent. Equivalent expressions look different but always have the same value.

EXPRESSIONS AND ALGEBARS

Algebars are bars that represent algebraic quantities. Red bars represent variables; green bars represent expressions.

1. Open **Equivalent Expressions.gsp** and press the *Show Variables* button. Two red algebars appear representing the variables *a* and *b*.
2. Press the *Show Algebars 1* button. Two green algebars appear, labeled with their algebraic expressions: $a+b$ and $b+a$.



If two bars are always equal in length, they represent equivalent expressions. Try many values of the variables before you decide that two expressions are equivalent.

Q1 Predict what will happen if you change *a* and *b* by dragging the tips of the red algebars. Will the green algebars remain equal in length? Drag *a* and *b* to find out. Were you right? Are the two algebraic expressions equivalent? If so, write the result as an equation.

3. Press the *Show Algebars 2* button. Two more green algebars appear, representing two different algebraic expressions.

Use the indicator lines to estimate the values of *a* and *b.*

Q2 Drag *a* and *b* again. Are the new algebars always equal in length? Are there any positions for *a* and *b* that make these bars equal? Describe them. Are the two algebraic expressions equivalent?

4. Press the *Show Algebars 3* button. Two more green algebars appear, representing two different algebraic expressions.

Q3 Drag *a* and *b* again. Are the new algebars always equal in length? Are the two algebraic expressions equivalent? If so, write your result as an equation.

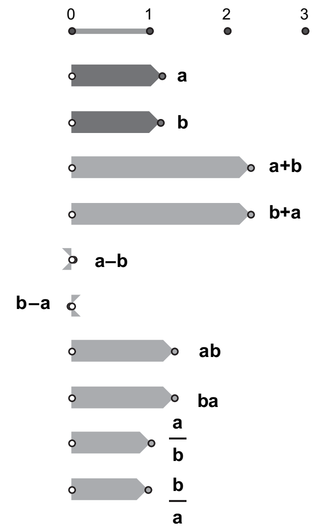
5. Press the *Show Algebars 4* button. Two more green algebars appear without their labels.

Q4 Drag *a* and *b* again. Are these bars always equal in length? If not, when are they equal? Do you think the algebraic expressions for these two bars are equivalent?

Q5 Press the *Show Expressions 4* button to see the labels for the last two bars. On your paper summarize your conclusions by filling in the blanks in a sentence like this: ____________ = ____________ when ____________.

THE COMMUTATIVE PROPERTY

If an operation is *commutative,* you can perform it in either order (for instance, $a + b$ or $b + a$) and get the same result.



Q6 Go to page 2. This page shows four possible commutative properties (for addition, subtraction, multiplication, and division). Predict which algebars will stay the same length when you drag *a* and *b*.

Q7 Drag *a* and *b* to test your prediction. Which pairs of algebars are always the same length? Write an equation for each pair that matches. Which of the four arithmetic operations are commutative?

THE DISTRIBUTIVE PROPERTY

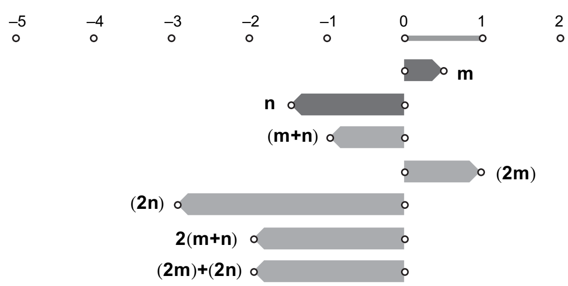
6. On page 3, drag *c* and note how the bars change.

Q8 Which two expressions are equivalent? Why?

Q9 Why is $2c + 4$ not equivalent to $2(c + 4)$?

Sabrina says that you can evaluate the expression $2(m + n)$ either the way it's written (add first, then multiply by 2) or by first multiplying the 2 by each of the values in the parentheses and then adding the results.

7. On page 4 are the bars Sabrina created to test her rule. Press the *Show Sabrina's Algebars* button to show them. The bottom two bars show the expression both the original way and Sabrina's way: $2(m + n)$ and $(2m) + (2n)$.



Q10 Drag *m* and *n* back and forth. Are $2(m + n)$ and $(2m) + (2n)$ equivalent? If so, write your result as an equation.

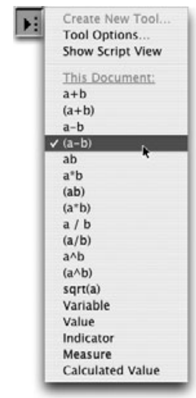
Corey says that you can do something similar with the expression $2 + (m \cdot n)$: You can first add the 2 to both values in the parentheses and then multiply the results.

8. Press the *Show Corey's Algebars* button. The last two bars show the expression both the original way and Corey's way: $2 + (m \cdot n)$ and $(2 + m) \cdot (2 + n)$.

Q11 Drag *m* and *n*. Are the expressions $2 + (m \cdot n)$ and $(2 + m) \cdot (2 + n)$ equivalent?

OTHER EXPRESSIONS

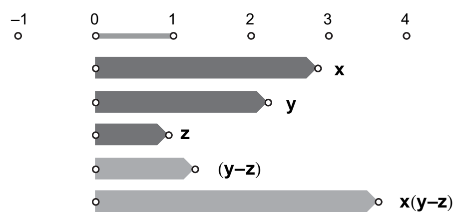
9. On page 5, build your own algebars to test whether $x(y - z) = xy - xz$. Start by constructing the expression $(y - z)$.



10. Press and hold the **Custom** tools icon and choose the **(a−b)** tool.

11. Click this tool on five objects: the top white point that's not already used, the point at the tip of the *y* algebar, the caption on the *y* algebar, the point at the tip of the *z* algebar, and the caption on the *z* algebar.

12. To finish constructing $x(y - z)$, choose the **ab** tool and click it on five objects: the starting white point, the tip and caption of the *x* algebar, and the tip and caption of the $(y - z)$ algebar.



Next construct the expression $xy - xz$. Start by constructing *xy* and *xz*.

13. Construct *xy* by using the **ab** tool on the *x* and *y* bars. Then construct *xz*.

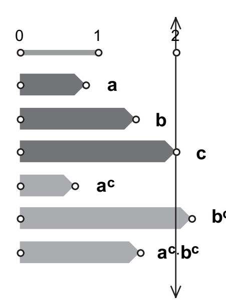
14. Construct $xy - xz$ by using the **a–b** tool on your *xy* and *xz* bars.

Q12 Drag *x*, *y*, and *z*. How does your test turn out? Does $x(y - z) = xy - xz$?

EXPLORE MORE

To change the label of a variable, select the point at the tip of the bar and choose **Display | Label Point.**

Q13 On page 6, build the expressions below. Use the **a^b** tool to raise a value to a power. Which of the three are equivalent? Write your answer as an equation.



$$a^c b^c \qquad (a + b)^c \qquad (ab)^c$$

Q14 Find values of *a*, *b*, and *c* that make all three of these expressions equal. What values of the variables did you use? How many sets of values can you find?

Q15 On page 7, test the following expressions to see if they are equivalent. Describe your conclusions.

$$y \qquad \left(\sqrt{y}\right)^2 \qquad \sqrt{y^2}$$

Equivalent Expressions

Objective: Students compare algebraic expressions and explore the equivalence of those expressions by using *algebars* (a model that shows the changing value of an algebraic expression by the varying length of a bar).

Student Audience: Algebra 1

Prerequisites: Students should know the term *expression* and have experience with order of operations.

Sketchpad Level: Intermediate. The first part is easy, but students use challenging custom tools by the end.

Activity Time: 30–40 minutes. Students can complete the main part of the activity in the time allotted, but the Explore More section may require a significant amount of additional time.

Setting: Paired/Individual Activity (use **Equivalent Expressions.gsp**)

By manipulating and constructing algebars, students explore the commutative and distributive properties and various rules involving exponents. Even more important, by dragging variables and observing the changes in the expressions based on the variables, students get used to the dynamic behavior that variables and expressions show. This sense that algebraic expressions are changeable, that they represent an entire range of possible values, is easier for students to internalize when they can actually see the values in motion.

This activity is also valuable preparation for other activities that use algebars (for instance, Undoing Operations and Solving Linear Equations by Undoing).

INVESTIGATE

Q1 Encourage students to make a prediction before dragging. As students drag a and b, the green algebars remain the same length, because $a + b = b + a$ is an example of the commutative property of addition.

Q2 As students drag a and b, these two bars are seldom equal in length. The expressions $a - b + 1$ and $b - a + 1$ are not equivalent, except when $a = b$.

Q3 As students drag a and b, these two bars are always equal in length. The expressions are equivalent:

$$a - (b - a) = a + (a - b)$$

Q4 These bars are the same length only when $b \geq 0$. Students must drag b to the left of 0 to discover that the two expressions are not equivalent.

Q5 $ab = a\sqrt{b^2}$ when $b \geq 0$.

Q6 Answers will vary. The important thing is that students make a prediction.

Q7 The addition algebars are always the same length, and so are the multiplication algebars. The equations are $a + b = b + a$ and $ab = ba$. The subtraction and division algebars are sometimes not the same length. Therefore addition and multiplication are commutative, and subtraction and division are not.

Q8 The expressions $2(c + 4)$ and $2c + 8$ are equivalent. As an equation, $2(c + 4) = 2c + 8$. This is an example of the distributive property of multiplication over addition.

Q9 Answers will vary. Some students may describe the behavior of the bars; others may give a counter-example; and others may give an algebraic argument in terms of the distributive property. The important thing is to get students to think about the question.

Q10 Yes, $2(m + n) = (2m) + (2n)$.

Q11 No, the expressions $2 + (m \cdot n)$ and $(2 + m) \cdot (2 + n)$ are not equivalent.

Q12 Yes, $x(y - z) = xy - xz$.

EXPLORE MORE

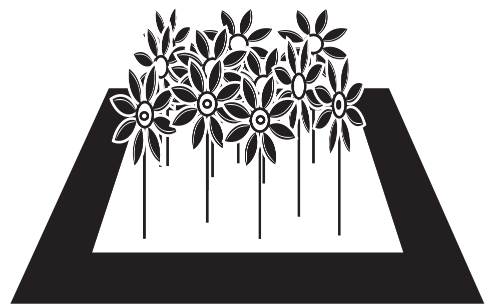
Q13 The expressions $a^c b^c$ and $(ab)^c$ are equivalent because exponents are distributive across multiplication, but not addition.

Q14 When $c = 0$, all of the expressions are equal for any values of a and b. If c is any other value, then students must experiment with values of a and b. A simple solution would be setting $a = b = 2$ since $2 \cdot 2$ and $2 + 2$ are equal. Other numerical values for a and b make the expressions equal when $a = b/(b - 1)$.

Q15 No two of these expressions are equivalent. All three give different results when $y < 0$: the result of the first is negative, the result of the second is undefined, and the result of the third is positive.

Equivalent Expressions: The Border Problem

Suppose you want to build a square flower bed surrounded by a 1 m wide border to serve as a path. You need to know the area of the border so that you'll know how much gravel to buy. You haven't decided how big a square the flower bed should be, but you want to be able to find the border's area for any flower bed with side length *s*. You need an algebraic expression.



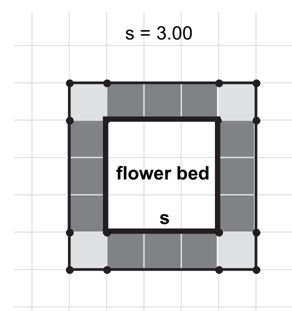
INVESTIGATE

Equivalent expressions are expressions that look different but that give the same result, such as $x - 2$ and $x + (-2)$.

Several different equivalent expressions are possible for this problem. Your goal is to come up with as many of them as you can and test them for equivalence by seeing if they all give the same area for any side length.

1. Open **Border Expressions.gsp.**

You'll see a square with a border around it, representing the flower bed and the path. Eight quadrilateral interiors are constructed on the border—four green and four yellow. This arrangement represents one possible expression for the area of the border: $4s + 4$.



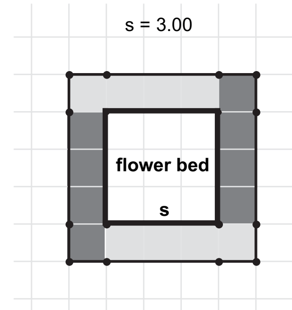
Q1 Explain how the expression $4s + 4$ relates to the quadrilaterals around the flower bed. Why does this expression give the area of the border?

If the Calculator covers the measurement of *s*, drag the Calculator aside.

2. Use Sketchpad's Calculator to find the value of $4s + 4$. To do this, choose **Calculate** from the Measure menu. Then type 4, click on the measurement for *s* in the sketch, add 4, and click OK.

Q2 Did you get the correct value for the area of the border? Drag the red point at the end of the *s* slider. Is the calculation still correct? Pick three different values of *s*. Show how substituting these values for *s* gives the correct value for the area of the border.

3. Go to page 2. You'll see another arrangement of quadrilaterals around the border.



Q3 What expression does this arrangement represent?

4. Again, use Sketchpad's Calculator to find the value of your expression. Do you get the correct value for the area even when you drag the red point?

5. Go to page 3. You'll see flower beds for the two previous arrangements, and many more with borders yet to be filled in.

To construct a quadrilateral interior, select its four vertices in order and choose **Construct | Quadrilateral Interior.**

6. On one of the empty flower beds, construct a different set of quadrilaterals that divide the border area in a new way to represent a different expression for the area.

7. Use Sketchpad's Calculator to calculate the value of your expression. Does your expression work even as you drag the red point to change the value of s?

8. Repeat steps 6 and 7 two more times.

Q4 Write your three new equivalent expressions from steps 6–8.

EXPLORE MORE

Q5 Find as many more equivalent expressions as you can.

Q6 Plot the functions associated with all of your equivalent expressions. How do the plots demonstrate their equivalence?

(For the expression $4s + 4$, for example, plot the function $f(x) = 4x + 4$. To do this, go to page 4, choose **Graph | Plot New Function,** enter $4x + 4$, and click OK.)

Q7 Use properties of algebra to prove that all of your expressions are equivalent.

Q8 Find an equivalent expression by calculating the area of the bed and subtracting it from the area of the border and bed combined. Simplify your result and show that it is equivalent to your other expressions.

Equivalent Expressions: The Border Problem

Objective: Students invent a variety of equivalent expressions for a real-world area problem.

Student Audience: Algebra 1

Prerequisites: Students need to know how to find the areas of squares and rectangles and have some basic facility with algebraic notation.

Sketchpad Level: Intermediate. Students construct interiors and perform calculations.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Border Expressions.gsp** in either setting)

INVESTIGATE

The concept of units isn't explored explicitly in this activity. If s is the side length of the square—and is therefore measured in linear units such as centimeters (cm)—how can s also represent the area of one of the yellow rectangles, which should use square units such as cm^2? The answer is that the area of the yellow rectangle is the product of its lengths, or $(s \text{ cm}) \cdot (1 \text{ cm}) = s \text{ cm}^2$ (assuming we're in centimeters). This is a good topic for class discussion.

Q1 The area of each yellow rectangle is $1s$, or simply s. The area of all four of these is then $4s$. The area of one of the green squares is 1 since they are 1×1 squares. The area of all four squares is then 4, and the total area of the border is $4s + 4$.

Q2 Sample answers:

$$4(1) + 4 = 8$$
$$4(2) + 4 = 12$$
$$4(3) + 4 = 16$$

Q3 $4(s + 1)$

The area of any one of the four rectangles is $s + 1$ since each extends 1 unit beyond the side length s. (In other words, each is a green rectangle plus a yellow square.) The area of all four rectangles is then $4(s + 1)$.

Q4 Many possible answers. Here are several:

$$2(s + 2) + 2s$$
$$s + s + s + s + 1 + 1 + 1 + 1$$
$$(s + 2) + 3s + 2$$
$$(s + 2) + 2(s + 1) + s$$
$$3(s + 1) + s + 1$$

EXPLORE MORE

Q5 See Q4.

Q6 The plots are all right on top of each other.

Q7 One way to approach this would be to show that all of the expressions are equivalent to $4s + 4$. By the transitive property, they are all then equivalent to each other. To prove that $4(s + 1)$ is equivalent to $4s + 4$, for example, use the distributive property.

Q8 The area of the border and bed combined is $(s + 2)^2$, or $s^2 + 4s + 4$. Subtracting the area of the flower bed, s^2, we once again find $4s + 4$ for the area of the border.

WHOLE-CLASS PRESENTATION

This activity works well as a whole-class activity with a presentation computer and projector. Describe the situation to the class and have students make the appropriate drawing. Then ask the class for an expression for the border area. Have a student who came up with the expression justify it with a drawing. Then open up **Border Expressions.gsp** and show them this solution in Sketchpad. Calculate the area expression and ask if it seems to give the right answer for various values of s. Then go to page 3 and have students come up with several more expressions on their own or in groups. One by one, have students (or possibly group representatives) build their expressions in Sketchpad and do the associated calculations. Finish up the activity with a discussion of the various expressions—why some worked and some didn't.

The Distributive Property: A Painting Dilemma

The school activities committee is preparing to paint two gymnasium walls. Both walls are 25 ft high. The first wall is 80 ft wide, and the other is 100 ft wide.

Cari and Zeeba are on the committee, and they volunteered to calculate how many cans of paint to order. To figure this out, they need to find the total area of the walls. But Cari and Zeeba disagree about how to do the calculation.

Cari wants to calculate the area of each wall separately and then add them together to get the total painted area:

$$(25 \text{ ft})(80 \text{ ft}) + (25 \text{ ft})(100 \text{ ft})$$

Zeeba wants to calculate the total area by first adding up the widths of the walls to get the total width and then multiplying the height by this total:

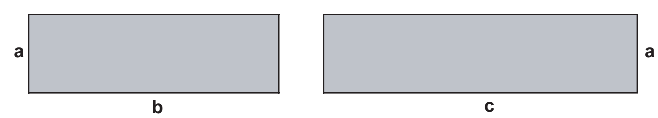
$$(25 \text{ ft})(80 \text{ ft} + 100 \text{ ft})$$

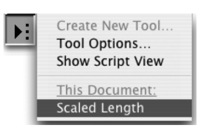
SKETCH AND INVESTIGATE

1. Open **Distributive Painting.gsp.** This is a perspective view of the two walls as seen from inside the gym.

First calculate the area using Cari's method.

2. Press the *Cari's Solution* button.





3. To measure the walls, press and hold the **Custom** tools icon and choose **Scaled Length.** Click this tool on line segments *a, b,* and *c.*
4. To calculate the area using Cari's method, first choose **Measure | Calculate** and find the area of the rectangle on the left. To enter the width and height of this wall into the Calculator, click the measurements in the sketch.

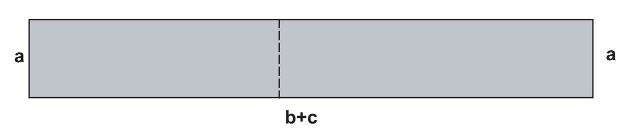
If you prefer, you can do all three calculations in one step.

5. Calculate the area of the rectangle on the right. Then finish calculating by Cari's method by using the Calculator one more time to find the sum of the two areas.

Q1 What is the result using Cari's method?

Now calculate the area using Zeeba's method.

6. Press the *Reset* button to return the walls to their original arrangement. Press the *Zeeba's Solution* button.



7. Measure the line segment labeled $b + c$.

8. Calculate the area using Zeeba's method, by using Sketchpad's Calculator to calculate the total area of this single rectangle.

Q2 What is the result using Zeeba's method?

Q3 Write your expressions as an equation using the variables *a*, *b*, and *c*. This equation is a symbolic statement of the *distributive property of multiplication over addition.*

You've tried your result for only one set of measurements. Jason claims that it might work for this one case, but it won't work for others.

9. To test Jason's claim, press the *Show Dimensions* button, change the length from 100 ft to 60 ft, and press *Reset.* Then press the buttons to try both Cari's solution and Zeeba's solution.

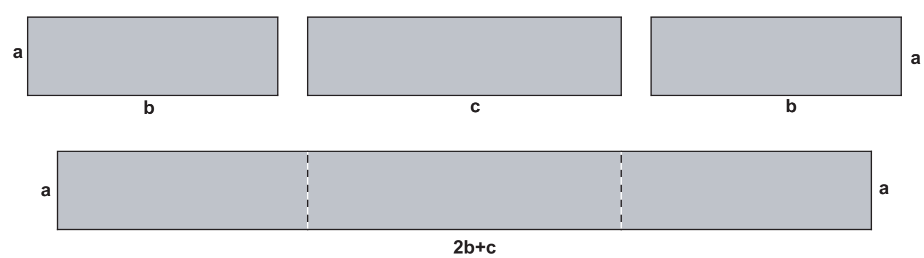
Q4 Do both methods still give the same result?

Q5 Make a conjecture. How would the distributive property apply if there were three walls, two with width *b* and one with width *c*?

THREE WALLS

10. Go to page 2 of the document.

11. After calculating the total area of the walls, the students find that they have enough money to paint a third wall in the gym. Repeat the previous steps and calculate the painted area using both Cari's and Zeeba's methods.



Q6 Do both methods give the same result? What is the combined area of the three walls?

Q7 Write your expression as an equation. Compare the equation with your conjecture in Q5.

Q8 It would be possible to measure the line segments while the sketch is showing a perspective view, but the results would not be very useful. Explain why this is true.

The Distributive Property: A Painting Dilemma

Objective: Students calculate an area to be painted two different ways and verify that both ways give the same result. They generalize the calculations to use variables instead of numbers, resulting in the distributive property for multiplication over addition.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: This activity requires an understanding of rectangle area calculations.

Sketchpad Level: Intermediate. Students use custom tools and the Calculator.

Activity Time: 15–25 minutes

Setting: Paired/Individual Activity (use **Distributive Painting.gsp**) or Whole-Class Presentation (use **Distributive Painting Present.gsp**)

SKETCH AND INVESTIGATE

Q1 The result for Cari's method is

$$\begin{aligned}&(25\text{ ft})(80\text{ ft}) + (25\text{ ft})(100\text{ ft})\\ &= 2000\text{ sq ft} + 2500\text{ sq ft}\\ &= 4500\text{ sq ft}\end{aligned}$$

Q2 The result for Zeeba's method is

$$\begin{aligned}&(25\text{ ft})(80\text{ ft} + 100\text{ ft})\\ &= (25\text{ ft})(180\text{ ft})\\ &= 4500\text{ sq ft}\end{aligned}$$

Q3 $ab + ac = a(b + c)$

Q4 Both methods still give the same result. For these particular numbers, the result is 3500 sq ft.

Q5 In the case of three walls, the factor a would be multiplied by the widths of all three walls:

$$ab + ab + ac = a(b + b + c) = a(2b + c)$$

THREE WALLS

Q6 Both methods do give the same result: 6500 sq ft.

Q7 See Q5.

Q8 Answers will vary. In the perspective view the scale is variable. This is because objects are foreshortened, and some objects are closer to the viewer than others. Moreover, a rectangle generally does not even appear as a rectangle.

WHOLE-CLASS PRESENTATION

Use **Distributive Painting Present.gsp** in conjunction with the Presenter Notes to present this activity to the whole class.

1. Open **Distributive Painting Present.gsp,** and explain the problem to your class based on the description at the beginning of the student activity sheet.
2. Press the *Cari's Solution* button. Explain that this is the configuration on which Cari decided to base her calculation.
3. Explain that the first part of Cari's method is to calculate the area of the rectangle on the left. Choose **Measure | Calculate** to show the Calculator. Click the 25 ft measurement in the sketch, the multiplication sign on the keypad, and the 80 ft measurement in the sketch. Click OK to finish the calculation.
4. Use a similar calculation to compute the area of the rectangle on the right. Then use the Calculator one more time to find the sum.

Q1 Ask, "How would you express this calculation using the variables *a, b,* and *c*?" (The label of the calculation gives the answer away. Congratulate students for noticing this.)

Now calculate the area using Zeeba's method.

5. Press the *Reset* button to return the walls to their original arrangement. Press the *Zeeba's Solution* button.
6. Use the Calculator to multiply the height (25 ft) by the length (180 ft). Be sure to click on the measurements in the sketch; don't type the numbers on the keypad.

Q2 Ask, "How would you express this calculation using the variables *a, b,* and *c*?" (The label of the calculation again gives the answer away. Don't congratulate the students this time.)

Q3 Ask, "Are the results equal?"

Q4 Ask, "Will the results always be equal, even if the dimensions are different?"

Even if the entire class says the results will always be equal, tell them that it's too easy to be fooled in problems like this and that it's worth trying different measurements.

7. Press the *Show Dimensions* button, double-click the *a* measurement, and change it to 20. Double-click the *c* measurement and change it to 60. Press *Reset.*
8. Press the *Cari's Solution* button to show Cari's result, and then the *Zeeba's Solution* button to show Zeeba's.

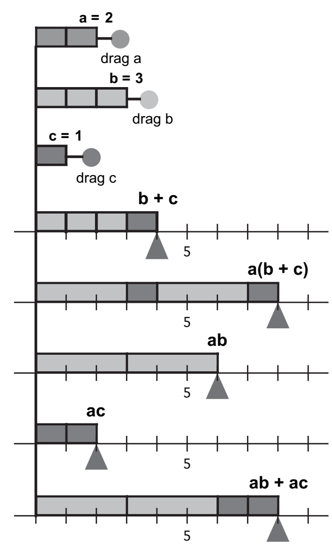
Q5 Ask students to summarize the principle of algebra that they've found by investigating this problem.

9. If there's time, explore page 2 (with three walls).

The Distributive Property

In the expression $a(b + c)$, the parentheses indicate that you must add the b and c first, and then multiply a by the result. But the distributive property says that you can distribute the a to the values in the parentheses, multiplying a by b and also multiplying a by c, and then adding the two results together.

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION



1. Open **Distributive Property.gsp.** Click the *Present Left Side* and *Present Right Side* buttons to show the two ways of calculating $a(b + c)$.

Q1 Write down the results of each step of the calculation.

2. Reset the sketch, drag the variables to represent $-3(1 + 4)$, and use the buttons to show the calculations again. Then try $2(-3 + (-1))$.

Q2 Explain in your own words how the animation demonstrates the distributive property and why the distributive property is true.

Q3 Drag the variables to represent $2(5 + (-3))$. Describe what happens when the rectangles overlap. Is the distributive property still true in this case?

Q4 Make up three more problems of your own. Use the sketch to show them, and write down the problems, the results, and anything interesting you observe.

EXPLORE MORE: OTHER DISTRIBUTIVE PROPERTIES

You can write the distributive property in a more general way, representing the two operations by $\Diamond$ and $\otimes$:

$$a \Diamond (b \otimes c) = (a \Diamond b) \otimes (a \Diamond c)$$

The last page of the sketch has directions for using these custom tools.

Q5 Rewrite this formula replacing $\Diamond$ with multiplication and $\otimes$ with addition. This is the normal distributive property you explored above. On page 2 use the **Add, Subtract, Multiply,** and **Divide** custom tools to calculate both sides of the formula and verify that the resulting bars are the same length.

Q6 Rewrite the formula replacing $\Diamond$ with addition and $\otimes$ with multiplication. Use page 3 to determine whether this modified distributive property is true.

Q7 Additional modified distributive properties are shown on pages 4 and 5. Verify or disprove each of them by constructing and comparing bars.

The Distributive Property

Objective: Students explore a graphic model of the distributive property and explain in terms of the model why the property is true. In Explore More, students test whether the property is still true if you interchange the role of addition and multiplication.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should be familiar with the order of operations. This activity can be used to introduce the distributive property or to reinforce the property after it has already been introduced.

Sketchpad Level: Easy. In the main part of the activity, students manipulate a pre-made sketch. The Explore More section requires students to use custom tools.

Activity Time: 20–30 minutes. Students can complete the main part of the activity fairly quickly but will need more time to do the Explore More section.

Setting: Paired/Individual Activity or Whole-Class Presentation (use **Distributive Property .gsp** in either setting)

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Q1 The left side is done in two steps: (a) Add $3 + 1$ to get 4, and (b) multiply $2 \cdot 4$ to get 8. The right side requires three steps because multiplying by 2 is distributed over the other two numbers: (a) Multiply $2 \cdot 3$ to get 6, (b) multiply $2 \cdot 1$ to get 2, and (c) add $6 + 2$ to get 8.

Q2 The animation for the very first example shows that either way of doing the problem results in 2 blocks of 1 and 2 blocks of 3, even though they are arranged differently. When these blocks are combined into an answer, the same blocks are being combined, so the result must be the same.

Q3 Even when one of the numbers being added is negative, the distributive property is still true. You just have to watch the blocks more carefully because some of the blocks are positive and some are negative, causing them to overlap.

Q4 Answers will vary.

EXPLORE MORE: OTHER DISTRIBUTIVE PROPERTIES

Q5 The formula becomes

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

This is the formula for the distributive property of multiplication over addition. The bars representing the left and right sides will always be of equal length.

Q6 The formula becomes

$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

This formula represents distributing addition over multiplication. The bars don't always match, so this formula is false in general (although students may find some specific numeric values for which it is true).

Q7 The formula on page 4 is the distributive property of division over addition:

$$a/(b + c) = (a/b) + (a/c)$$

The bars do not match, and this formula is false.

The formula on page 5 is similar, except that the division sign is on the right rather than the left:

$$(a + b)/c = (a/c) + (b/c)$$

These bars always match, and the formula is true.

Discuss why the formula on page 4 (called the left distributive property of division over addition) is false, but the one on page 5 (the right distributive property of division over addition) is true.

If students don't bring it up in the discussion, ask whether there's any relationship between the formula on page 5 and the formula they explored on page 2. The connection involves using multiplicative inverses and the commutative property of multiplication, as shown here:

$$(a + b)/c = a/c + b/c$$

$$(a + b) \cdot \frac{1}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$$

$$\frac{1}{c} \cdot (a + b) = \frac{1}{c} \cdot a + \frac{1}{c} \cdot b$$

WHOLE-CLASS PRESENTATION

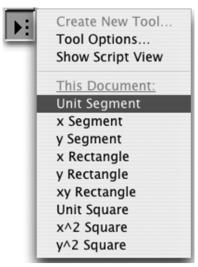
Use page 1 of **Distributive Property.gsp** to demonstrate the property and stimulate a discussion as to why the two expressions are equal. Use the remaining pages to investigate other possible distributive properties.

Algebra Tiles

In the development of mathematics, geometry came long before algebra. Scholars valued geometry because they could see it and draw it. Numbers by themselves were hardly more than rumors. Today, we use numbers (including variables) much more freely—but do not dismiss the lessons of the ancients. Geometry still provides us a means to lay the numbers out before our eyes. In this activity you will use algebra tiles to model numbers as segment lengths and rectangle areas.

LINE SEGMENT TOOLS

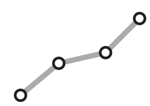
1. Open **Algebra Tiles.gsp.**



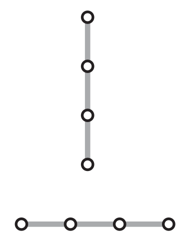
2. Press and hold the **Custom** tools icon to display the Custom Tools menu. Choose the **Unit Segment** tool. Click several times on the screen. Then choose the **Arrow** tool.

Q1 Each of these new line segments represents the unit value (1). Describe in detail what you see. Drag the endpoints and describe the resulting behavior.

3. Choose the **Unit Segment** tool again. Construct one unit segment. With the tool still active, click the black endpoint. Then click the black endpoint of the second segment.



Q2 You should now have three unit line segments end-to-end. At first, it is nothing but a scribble, with no particular number association. Drag the black points to align the parts into a single line segment, oriented vertically or horizontally. What number does its length represent?



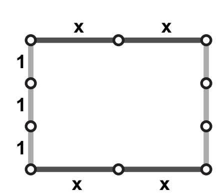
4. Choose the **x Segment** custom tool. Click in several places on the screen. Choose the **y Segment** custom tool and create several of those segments.

Q3 Describe the similarities and differences between the line segments on your screen.

Q4 Use the tools to construct a horizontal line segment of length $2x + 3$. Construct another one of length $x + 3y + 1$. Sketch and describe the images on paper.

Start at the lower left, and attach segments to each other by clicking the black endpoint of an existing segment.

Q5 Construct a rectangle like the one shown here. What is the area of the rectangle?



Q6 Drag the point at the end of the x slider. What happens in the sketch? Does the figure hold together?

AREA TOOLS

When you multiply the lengths of adjacent sides of a rectangle, the product is the area of the rectangle. You have been using line segment lengths to represent numbers. Now you will use rectangle areas to represent products of numbers.

5. Go to page 2. Choose the **x Rectangle** custom tool. Create a rectangle on the screen. Drag the black point so that it is in the lower-right vertex of the rectangle.

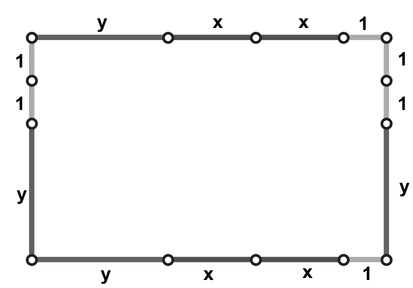
Q7 What are the width, height, and area of this rectangle? What happens when you drag the black point?

Each rectangle side has a hidden label indicating its value. You can show these labels by clicking the side with the **Text** tool.

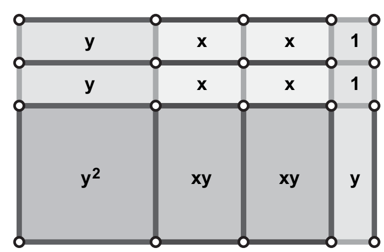
6. There are other tools, named **y Rectangle, xy Rectangle, Unit Square, x^2 Square,** and **y^2 Square.** Try each of them at least once.

EXPRESSIONS FOR AREA

The rectangle on the right is formed by linking line segments end-to-end. Its height is $y + 2$, and its width is $y + 2x + 1$. That makes its area $(y + 2)(y + 2x + 1)$.



Below that is the same rectangle again. This time its interior is tiled with smaller rectangles. By adding the areas of each of the smaller rectangles, you can get another expression for the area: $y^2 + 2xy + 3y + 4x + 2$.



These are two ways of expressing the area of the same rectangle, so the expressions must be equal. That fact leads to this equation:

$$(y + 2)(y + 2x + 1) = y^2 + 2xy + 3y + 4x + 2$$

Q8 Go to page 3. You will see four composite rectangles. For each rectangle, write an equation showing the area as the product of its height and width (factored form), and as the sum of the areas of the interior rectangles (expanded form).

Q9 Page 4 contains four more rectangles, but the interiors are not yet filled. Use the custom tools to tile the interiors. For each rectangle, write an equation with the factored and expanded forms of the area.

Objective: Students use custom tools to model expressions as line segment lengths and rectangle areas. They then use dissected rectangles to see the equivalence of expressions in factored and expanded form.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should learn symbolic expansion before doing this activity, but factoring is not required.

Sketchpad Level: Intermediate. The activity involves extensive use of several custom tools.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Algebra Tiles.gsp**) or Whole-Class Presentation (use **Algebra Tiles Present.gsp**)

You may consider creating your own activities by using the custom tools in **Algebra Tile Tools.gsp.**

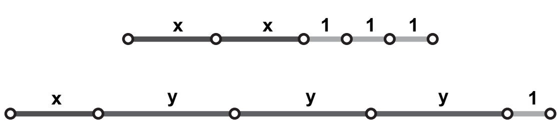
LINE SEGMENT TOOLS

Q1 All the unit line segments have the same length. Dragging the white endpoint translates the segment. Dragging the black point rotates it between vertical and horizontal orientations.

Q2 When the three line segments are in alignment, the figure represents the value 3.

Q3 The x line segments are blue. They are equal to each other in length, but longer than the unit line segments. The y line segments are longer still. They are red and are equal to each other in length. All of the line segments have the translation and rotation properties described in the answer to Q1.

Q4 To model $2x + 3$, construct two x segments and three unit segments end-to-end. To construct $x + 3y + 1$, do the same with one x segment, three y segments, and one unit segment. In both cases, the component line segments must be collinear.



Q5 The area of the rectangle is $6x$.

The task in Q5 gives students a chance to get used to the tools. The best way to construct this rectangle is to begin at the lower-left vertex, attaching each new segment to the endpoint of a previously constructed segment. First work your way across the bottom and up the right side. Then go back to the starting point and go up the left side and across the top. Straighten the sides by dragging the black points.

Q6 When students drag the x slider, the lengths of the x segments change. If students have attached the segments to each other, the figure holds together.

AREA TOOLS

Q7 The width is x and the height is 1. The black point rotates one side of the rectangle through 90°. If the black point is moved to the top of its path, the width becomes 1 and the height x. Even as the rectangle rotates, the lower left vertex stays in place.

EXPRESSIONS FOR AREA

Q8 a. $3(y + 1) = 3y + 3$

b. $(x + 2)(x + y) = x^2 + xy + 2x + 2y$

c. $(2y + 1)(y + x + 2) = 2y^2 + 2xy + 5y + x + 2$

d. $(x + y)(3x + 1) = 3x^2 + 3xy + x + y$

Q9 a. $(2x + 1)(3 + x) = 2x^2 + 7x + 3$

b. $(y + x + 1)(2y + x) = 2y^2 + 3xy + x^2 + 2y + x$

c. $(1 + 2x)(3x + 2y) = 6x^2 + 4xy + 3x + 2y$

d. $(2 + x)(4 + y) = xy + 4x + 2y + 8$

In Q9, students should start the constructions in the lower-left corner and work their way up and right. The rectangles will fit better (perfectly, in fact) if you click on an existing point when constructing each new rectangle. However, it is possible to construct each part independently and drag them into their approximate places. Two rectangles will fit together only if their common sides have the same color, hence, the same value.

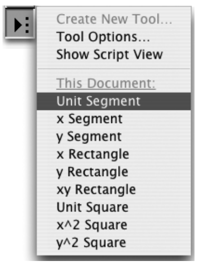
WHOLE-CLASS PRESENTATION

To present this activity to the whole class, use the Presenter Notes and the sketch **Algebra Tiles Present.gsp.**

In this presentation you'll show your class how dynamic algebra tiles work and use algebra tile diagrams to write two different expressions for the same area.

1. Open **Algebra Tiles Present.gsp.** The sketch is empty except for sliders representing 1, *x*, and *y*.

The lengths of the sliders will be used as the dimensions of the algebra tiles. Because the sliders are adjustable, the tiles will also be adjustable in size.



2. Press and hold the **Custom** tools icon to display the Custom Tools menu. Look at the list of tools with your students, and ask them to guess what the various tools might do.
3. Choose the **Unit Segment** tool and click several times on the screen.

Q1 Choose the **Arrow** tool and drag a segment, first by one endpoint and then by the other endpoint. Ask students what they observe about the segments.

4. Demonstrate the use of a different segment tool, one of the rectangle tools, and one of the square tools.
5. Show how to use the **Arrow** tool to drag the black vertex of a rectangle tile to orient the tile either horizontally or vertically.
6. Change the lengths of the unit, *x*, and *y*. Ask students to describe the effect in the sketch as you change these lengths.
7. On page 2 of the sketch are several constructions in which tiles have been assembled into rectangles. Press the *Show a* button to see the first rectangle.

Q2 Ask, "What is the height of this rectangle? Does the height use unit distances, *x* distances, or *y* distances? What is the width?" Encourage discussion. (The rectangle is 3 units high and $y + 1$ units wide.)

Q3 Ask students to use the height and width to write a formula for the area.

Q4 Ask, "What is the area of each small green square? And what is the area of each pink rectangle?" Ask students to add the areas to find the area of the rectangle.

Q5 Ask students what the relationship is between the two expressions for the area.

Look at Q8 and Q9 in the Activity Notes for the answers to these area problems.

Q6 Show figure *b* and have students find its area, both by multiplying and counting. Make an equation from the two expressions. Do the same for the remaining problems on page 2, and then do the animated problems on page 3.

Have students summarize their conclusions about the two ways in which they found the areas of the rectangles.

The Product of Two Binomials

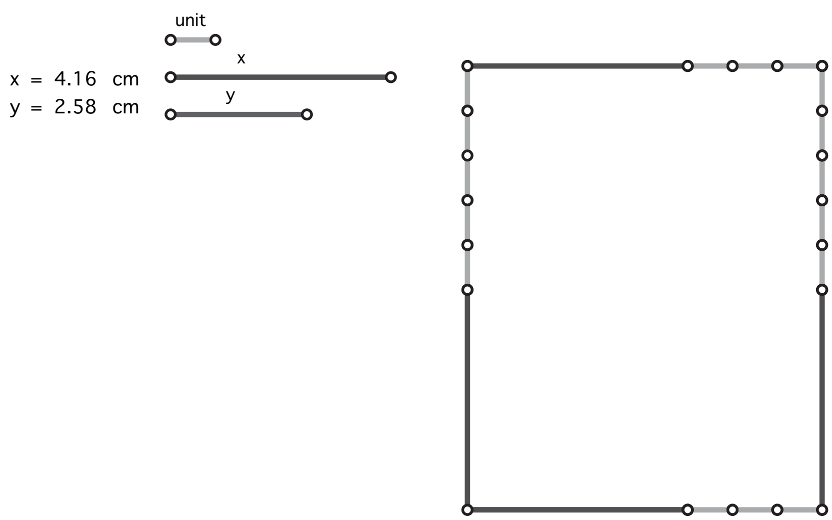
Mono-, *bi-*, and *tri-* are prefixes from the Greek words for one, two, and three, respectively.

The expression $x + 3$ is called a *binomial* because it consists of two *monomial* terms: x and 3. The expression $(x + 3)(x + 5)$ is the product of two binomials, $x + 3$ and $x + 5$. In this activity you'll use Sketchpad algebra tiles to model expressions equivalent to the products of binomials. The process you'll learn, called *expanding*, is used for writing expressions in different forms and for demonstrating the equivalence of algebraic expressions.

INVESTIGATE

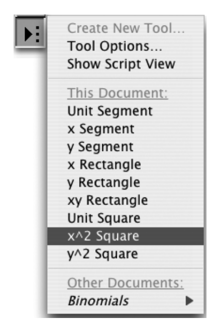
1. Open **Binomial Product.gsp.**

You'll see the factored expression $(x + 3)(x + 5)$ modeled with algebra tiles. The blue segments represent x and the green segments represent one unit. Notice that one binomial factor is modeled vertically and the other is modeled horizontally.



Press and hold the **Custom** tools icon to see the available tools. Choose the tool you wish to use, then click on a point in the sketch to construct that item. You can drag the black vertex later to rotate the shape to the orientation you want.

2. Use the custom tools that come with the sketch to tile the rectangle, using tiles whose dimensions match the horizontal and vertical segments.



When you're done, adjust the x slider and see if your rectangle holds together.

The expression you modeled is $x^2 + 5x + 3x + 15$. You can combine the like terms to get $x^2 + 8x + 15$. This *trinomial* (an expression with three monomial terms) is called the *expanded form* of $(x + 3)(x + 5)$.

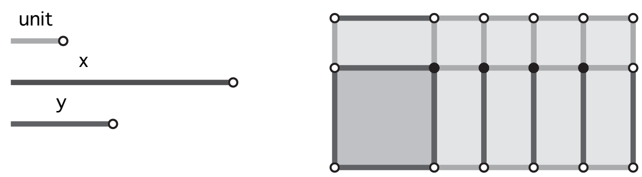
Q1 Explain how each of the terms in the trinomial $x^2 + 8x + 15$ is related to the product of the binomials.

Choose **Measure | Calculate** to open the Calculator. Click on the *x* measurement in the sketch, and type from your keyboard to build the expressions.

3. Use Sketchpad's Calculator and the measurement for x to calculate the values of the expressions $(x + 3)(x + 5)$ and $x^2 + 8x + 15$ for the current value of x.
4. Change the length of the x slider to confirm that the expressions remain equivalent for different values of x.
5. Go to page 2. Use the **y Segment** and **Unit Segment** custom tools to build the sides of a rectangle representing the product $(y + 1)(y + 4)$. Attach each segment to the black endpoint of the previous segment.

To orient the segments horizontally or vertically, drag the black endpoints with the **Arrow** tool.

6. Straighten the sides of your rectangle using the **Arrow** tool.
7. Use the various square and rectangle custom tools to tile your rectangle. Drag the y slider to make sure your rectangle correctly represents this area.



Q2 Write the expanded expression represented by the rectangle. Combine like terms.

8. Use Sketchpad's Calculator and the measurement for y to calculate the values of the expression $(y + 1)(y + 4)$ and the expanded expression you just found. Adjust the y slider to confirm that these expressions are always equivalent.

Q3 Build and expand the following expressions on the remaining pages of the sketch. Draw the models on paper. Write each expression both as a product of binomials and as a trinomial.

a. $(x + 2)(x + 3)$ b. $(2y + 1)(y + 3)$ c. $(x + y)(x + 2)$
d. $(x + 2)(x + 2)$ e. $(2x + y)(x + 2y)$ f. $(3y + 1)(2y + 2)$

EXPLORE MORE

Q4 Experiment with ways the model can be altered to represent expressions with negatives. For example, how could you represent $(x + 2)(x - 3)$? Illustrate and explain any models you think of.

Objective: Students use Sketchpad algebra tiles to model the expanding of binomials. Students then use the results to demonstrate the principles of equivalent expressions.

Student Audience: Algebra 1

Prerequisites: Students need to know how to find the areas of rectangles and have some basic facility with algebraic notation.

Sketchpad Level: Intermediate. Students make calculations based on measured values and use custom tools. This activity will be easier if students have already completed the Algebra Tiles activity.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Binomial Product.gsp**)

It may take the students a few minutes to get the hang of placing new tiles. When using the custom tools to create tiles, it is very useful to click on an existing point in the sketch (as opposed to clicking in blank space or on some other type of object). This serves to anchor the new tile. The point you wish to click on will be highlighted when the tool is positioned properly; when you see the point highlighted, it's time to click. If you make a mistake, just choose **Edit | Undo.** Improperly constructed tiles will not stay in alignment when you adjust the sliders.

It's important that students drag the sliders for x and y periodically—this is the big advantage of using Sketchpad algebra tiles, after all. First, dragging tests whether they've used the custom tools properly. But more importantly, it reinforces the fact that x and y are variables and that the relationships discovered work no matter what their values.

INVESTIGATE

Q1 The term x^2 refers to the big square in the lower left of the rectangle you built inside the frame. It's the product of the x from $(x + 3)$ and the x from $(x + 5)$.

The term $8x$ refers to all of the non-square tiles: $5x$ from the upper left and $3x$ from the lower right. The $5x$ is the product of the x from $(x + 3)$ and the 5 from $(x + 5)$. The $3x$ is the product of the x from $(x + 5)$ and the 3 from $(x + 3)$.

The number 15 refers to the 15 unit squares at the upper right of the rectangle. It's the product of the 3 from $(x + 3)$ and the 5 from $(x + 5)$.

5. When students use a segment tool to construct a new segment, the segment is anchored where they click, but its direction is not yet determined as horizontal, vertical, or in between. To orient the segment vertically or horizontally, students must use the **Arrow** tool to drag the black point at the other end of the segment.

   Some students may prefer to straighten each segment as soon as they construct it. Others may prefer to attach several segments at various angles and then straighten them all later.

Q2 $y^2 + 5y + 4$

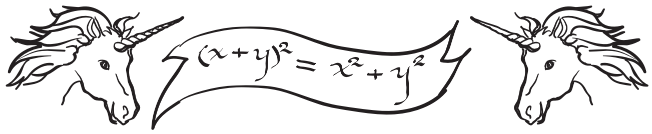
Q3
a. $x^2 + 5x + 6$
b. $2y^2 + 7y + 3$
c. $x^2 + xy + 2x + 2y$
d. $x^2 + 4x + 4$
e. $2x^2 + 5xy + 2y^2$
f. $6y^2 + 8y + 2$

EXPLORE MORE

Q4 Possible answers: Use an "opposite" color to represent negatives, or use dashed lines or shading. Students will come to appreciate the difficulty of representing negatives (particularly negative areas).

Squaring Binomials

Ask an algebra teacher what the most common mistakes are among algebra students. This one is certain to be near the top of the list:



In general, this equation is not correct. There is no property that allows you to distribute exponents over addition this way. Yet even the best students continue to make this mistake years after they have learned better. The fallacy of this equation becomes clearer when you model it geometrically with the help of algebra tiles.

TILING THE SQUARE

1. Open **Squaring Binomials.**

Use the **Arrow** tool to drag the endpoints of the sliders.

Q1 Drag the *x*, *y*, and *unit* sliders. What effect do they have on the square?

The sides of the square are $x + y$, so its area must be $(x + y)^2$.

2. Press and hold the **Custom** tools icon. Choose the **x^2 Square** tool from the menu that appears. Click in an open place on the screen to create the square.

Q2 In terms of *x* and *y*, how long is each side of the new square? What is its area?

3. Choose the **y^2 Square** custom tool and use it to construct another square in open space.

Q3 In terms of *x* and *y*, how long is each side of this square? What is its area? What is the sum of the areas of the two squares you just constructed?

If the equation $(x + y)^2 = x^2 + y^2$ is correct, then the area of the outlined square (on the left side of the equation) must be equal to the sum of the areas of the two new squares you just constructed (on the right side of the equation).

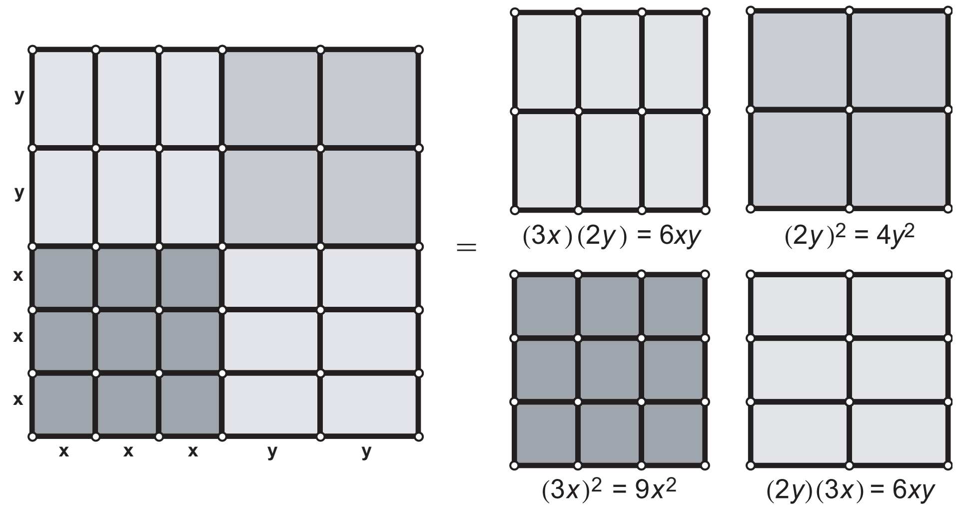
Q4 Test whether the equation is true by dragging the two new squares into position so that they precisely cover the outlined square. Can you do this? Explain your answer.

Each of the *xy* rectangle tiles has one black vertex. Drag this point to rotate the rectangle.

4. In fact, $(x + y)^2 = x^2 + 2xy + y^2$, so you will need more tiles to fill the square. Choose the **xy Rectangle** custom tool. Click twice on the screen. Choose the **Arrow** tool again.

Q5 Is it possible to fill the large square with the tiles you have created? Move the tiles into place to fill it. Draw a diagram showing how you did this.

What if you square a more complicated binomial? Consider $(3x + 2y)^2$. You can see four parts of the square separated in the image on the right. By counting shapes, you can see that $(3x + 2y)^2 = 9x^2 + 12xy + 4y^2$.



Q6 Now try it with a different binomial. Go to page 2. There is a square representing $(4x + y)^2$. Use the custom tools to form four groups of tiles, as shown in the preceding example. Then drag the tiles into the large square to fill it. Draw a diagram of the results and write the equation they represent.

BINOMIALS WITH SUBTRACTION

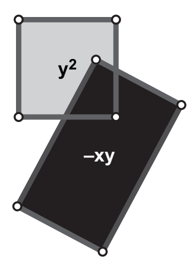
What if the binomial has a subtraction sign separating the two terms? In that case, the middle term of the expansion is negative:

$$(x - y)^2 = x^2 - 2xy + y^2$$

In order to show this geometrically, you need a way to show negative area.

The black rectangles represent negative area.

5. Go to page 3 of the document. There are light-colored squares and two black rectangles.



To add area, put two shapes next to each other. To subtract area, cover up a positive area with a negative area.

6. Add x^2 and y^2 by putting their squares next to each other. Then subtract $2xy$ by covering as much of the positive area as you can with the $-xy$ rectangles.

Q7 How much positive area is left? How does this remaining area compare with the $(x - y)^2$ square? Draw a diagram of your results from step 6.

Objective: Students use algebra tiles to model the identities $(x + y)^2 = x^2 + 2xy + y^2$ and $(x - y)^2 = x^2 - 2xy + y^2$.

Student Audience: Algebra 1

Prerequisites: This activity would be best shortly after students have learned to expand the square of a binomial.

Sketchpad Level: Intermediate. Students use custom tools. If they have not performed the Algebra Tiles activity, they will need more guidance.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Squaring Binomials.gsp**) or Whole-Class Presentation (use **Squaring Binomials Present.gsp**)

TILING THE SQUARE

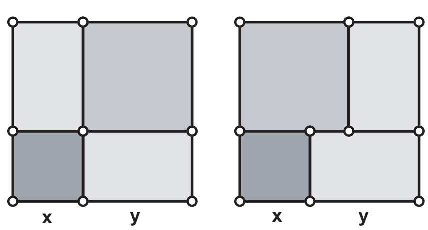
Q1 The *x* and *y* sliders change the sides of the square.The *unit* slider does not. This makes sense, because the sides of the square are defined in terms of *x* and *y* only.

Q2 The side length of this square is x. (Students can test this by dragging it next to the *x* slider.) The area is x^2.

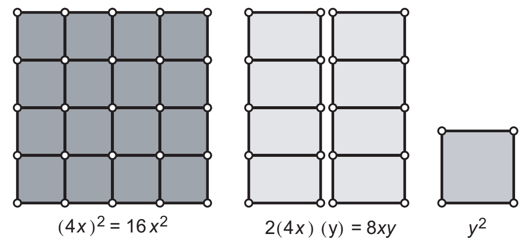
Q3 The side length of this square is y. (Students can test this by dragging it next to the *y* slider.) The area is y^2. The sum of the areas of these two squares is $x^2 + y^2$.

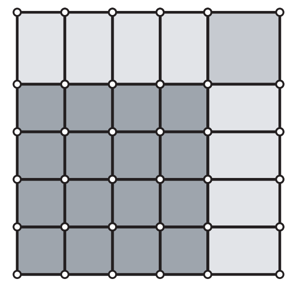
Q4 Generally, this is not possible. There is extra space in the outlined rectangle that cannot be covered by the two smaller tiles. Some students may discover a special case. It is, in fact, possible if x or y is equal to zero. They can model this case by changing the control sliders.

Q5 These tiles will fill the square no matter what lengths are used for x and y. Two possible solutions are shown below:



Q6 Below is one pattern that will work:

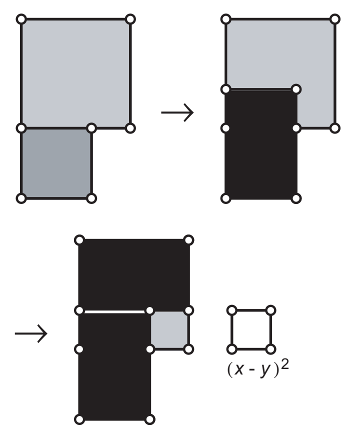




The equation is $(4x + y)^2 = 16x^2 + 8xy + y^2$.

BINOMIALS WITH SUBTRACTION

Q7 If the squares are covered as shown below, the only light color remaining will be a square with area $(x - y)^2$.



WHOLE-CLASS PRESENTATION

To present this activity to the whole class, use the Presenter Notes and the sketch **Squaring Binomials Present.gsp.**

In this presentation students will see a visual representation of the squaring of a binomial, and will make a connection between the squares and rectangles on the screen and the various terms that make up the algebraic expression.

1. Open **Squaring Binomials Present.gsp.** The empty square represents the square of the binomial $x + y$.
2. Change the value of x by dragging the point at the end of the blue slider.

Q1 Ask, "What happens when the value of x changes? How does it affect the two colored squares? How does it affect the rectangles?" (Only one square, representing x^2, is affected, and only the x dimension of the rectangle changes.)

Be sure to leave the values of x and y significantly different.

3. Similarly, change the value of y and have students observe the effects.
4. Drag the colored shapes into the empty square and arrange them so they fill the empty square. To tilt a rectangle to the correct angle, drag its black vertex.

Q2 Ask, "Geometrically, the shapes fit exactly. What does this mean algebraically?" Have students describe the connection between the shapes and the four terms on the right-hand side of $(x + y)^2 = x^2 + xy + xy + y^2$.

If you use the custom tools to attach the shapes, they will fill the square even when you change x and y.

5. Drag x or y to change the size of the empty square. The pieces are no longer in the correct positions. Press the *Tile* button to move the shapes so they fit again. Make sure students are convinced that the shapes will always fit.
6. Go to page 2, representing $(4x + y)^2$. Drag x and y to change the shapes. There are lots of tiles to move here, so use the *Tile* button right away.

Q3 Ask students to count the shapes and to write a formula based on the number of squares and rectangles they count: $(4x + y)^2 = 16x^2 + 8xy + y^2$.

The black rectangles can only be used to cover up positive area.

7. Go to page 3. This page represents $(x - y)^2$. Drag x and y to show how the shapes change. Point out that the black $-xy$ rectangles represent negative area.
8. Add the x^2 and y^2 tiles by placing them next to each other such that one vertex and one side coincide. Drag the $-xy$ rectangles so they cover as much of the positive area as possible.

Q4 Ask, "How much positive area is left after the subtraction?" Students may guess that the remaining positive area is equal to the $(x - y)^2$ square. Confirm this by dragging the $(x - y)^2$ square so it coincides with the remaining positive area.

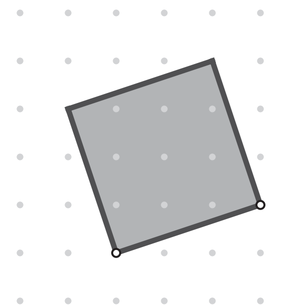
9. Change the values of x and y, switching which of them is larger. Separate the tiles, and then use the *Tile* button to arrange them again.

Q5 Ask students to write the formula that the tiles illustrate: $(x - y)^2 = x^2 + y^2 - xy - xy$.

Squares and Square Roots

Dot paper is graph paper on which the grid is made up of dots instead of intersecting lines. Dot paper is useful for exploring math and creating art.

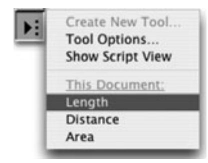
In this activity you'll use Sketchpad dot paper to explore several interesting properties of squares and square roots.



INVESTIGATE

1. Open **Square Roots.gsp.** You'll see a square constructed on dot paper. Drag one of the red corner points to change the size and orientation of the square.

Press and hold the **Custom** tools icon to choose one of these tools. Use the **Length** tool by clicking on a segment. Use the **Area** tool by clicking on the polygon interior.



2. Use the **Length** custom tool to measure the length of a side of the square, and use the **Area** custom tool to measure the area of the square. Drag one of the points and watch the measurements change.
3. Calculate the square root of the area. To do this, choose **Measure | Calculate** to open Sketchpad's Calculator. Choose **sqrt** from the Functions pop-up menu, click on the area measurement in the sketch, and click OK.

Q1 What do you notice about the value of this calculation? Drag the points to make sure your observation is true for different squares. Complete these two statements:

The area of a square is the _____________ of its side length.

The side length of a square is the _____________ of its area.

Q2 Given what you know about squares, why do the relationships in Q1 make sense?

Q3 Use your sketch to find the square roots of 12 whole numbers less than or equal to 20. Round all decimals to two places.

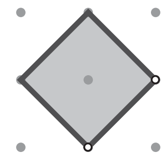
n												
$\sqrt{n}$												

Q4 Do you think it's possible to find the square root of any desired whole number using the method from Q3? Explain your reasoning.

PATTERNS

You can find many interesting number patterns in this sketch.

Q5 Set up the square so that it's perfectly balanced on its tip—in other words, so its diagonals are horizontal and vertical. The smallest such square has an area of 2. The next has an area of 8. You can think of this as a number sequence: 2, 8, ... Use your sketch to determine the first eight terms of this sequence. Write them down. Then describe the numeric rule for the sequence.

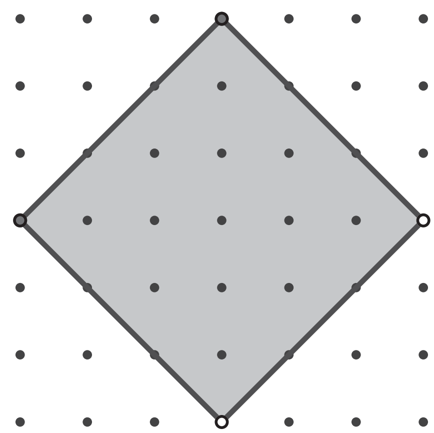


Pythagorean triples are numbers that satisfy the Pythagorean theorem: $a^2 + b^2 = c^2$.

Q6 When the square's base is horizontal, its side length is a whole number. But when the square is "slanted," its side length is usually an irrational number. In fact, you'll find only six slanted squares whose side lengths are whole numbers and that have at least one side completely within the sketch window. What are the side lengths of three such squares? (*Hint:* Think about Pythagorean triples.)

EXPLORE MORE

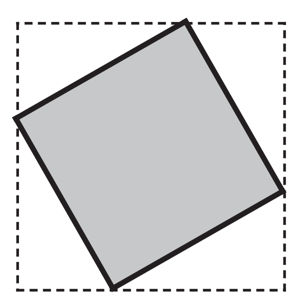
Q7 In Q5, you looked at a number pattern in the areas of squares as a vertex point is dragged. Draw the first four squares on grid paper or graph paper. Show how the area of the first square can be rearranged into 2 square units, the second square into 8 square units, and so on.



Q8 Look again at the squares and areas from Q5. The side lengths of the squares are the square roots of the areas. Write the sequence of square roots and simplify each one if possible. (To simplify, write each area as the product of its factors. For instance, $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.) Describe the numeric rule for this sequence. Explain how it relates to the numeric rule from Q5 and why this makes sense.

Q9 Find another number pattern as you drag a vertex in a systematic way, and write a numeric rule for that pattern.

Q10 You may have noticed that even slanted squares have whole-number areas. Write an argument explaining why. The figure at right may provide a hint.



Squares and Square Roots

Objective: Students use Sketchpad dot paper to explore properties of squares and square roots.

Student Audience: Algebra 1

Prerequisites: None. Some familiarity with rational numbers, irrational numbers, and the Pythagorean theorem helps.

Sketchpad Level: Easy

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Square Roots.gsp**)

An important concept that this activity touches on but does not cover in depth is rational versus irrational numbers. You may want to start the activity by giving students a little background on this topic. Interesting questions to ask (once the definitions are understood) are: "Could the square of an irrational number be a rational number?" "Could the square root of an irrational number be a rational number?" and "Could a square have an irrational side length but a rational area?"

INVESTIGATE

Q1 The area of a square is the *square* of its side length.

The side length of a square is the *square root* of its area.

(This is why taking something to the second power is called "squaring" and why "square root" is called what it is.)

Q2 The formula for the area of a square is $A = s \cdot s$ (where s refers to the length of a side). Multiplying something by itself *is* squaring; thus, area is the square of side length. Similarly, finding out what number times itself gives a certain result is finding the square root; thus, side length is the square root of area.

Q3 There are *exactly* 12 whole numbers less than or equal to 20 whose areas can be represented on a grid, or 13 if you include 0. The completed table should contain 12 of the 13 entries listed here.

n	0	1	2	4	5	8	9	10
$\sqrt{n}$	0	1	1.41	2	2.24	2.28	3	3.16

n	13	16	17	18	20
$\sqrt{n}$	3.61	4	4.12	4.24	4.47

Q4 No; students will have seen that some whole numbers can't be modeled as the areas of squares on a grid. Additional trial and error will show that some areas, such as 7, cannot be modeled with a square on this grid.

PATTERNS

Q5 2, 8, 18, 32, 50, 72, 98, 128, ... Dividing each term by 2 will reveal the sequence of perfect squares. The rule is $2n^2$ where n is the term number. Some students may see instead the pattern of differences between terms: 6, 10, 14, 18, and so forth.

Q6 Students may list any three of these answers; each is the third number of a Pythagorean triple:

5 (3 − 4 − 5)	15 (9 − 12 − 15)
10 (6 − 8 − 10)	17 (8 − 15 − 17)
13 (5 − 12 − 13)	20 (12 − 16 − 20)

EXPLORE MORE

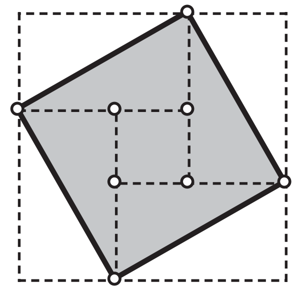
Q7 Different rearrangements are possible, but one way is to use the diagonals to divide the square and rearrange the four resulting triangles into two squares. (This also illustrates why the area rule is $2n^2$, since the area of each resulting square is n^2.)

Q8 $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \sqrt{50}, \sqrt{72}, \sqrt{98}, \sqrt{128}, \ldots$ which can be simplified to $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}, 6\sqrt{2}, 7\sqrt{2}, 8\sqrt{2}, \ldots$. The numeric rule is $n\sqrt{2}$, which is the square root of $2n^2$. This makes sense because the side length of a square is the square root of the area.

Q9 Answers will vary. Sample answer: Start with an unslanted square of area 9 sq. units and with the red vertices at the bottom. Drag either red vertex vertically upward one unit at a time. The area of the square increases in this sequence: 10, 13, 18, 25, 34, 45, ... The rule is $n^2 + 9$.

Q10 Because the vertices of the square are on grid points, the corners of the outer dotted square must also be on grid points. Therefore, the area of the outer square is a whole number. Similarly, the outer triangles have whole-number sides and can be combined to form two rectangles that must also have a whole-number area. The area of the inner square is the difference between these two areas, so subtracting a whole-number area from another whole-number area must give a whole-number result for the area of the square.

Rather than subtracting, some students might prefer a dissection approach:



The dissection lines are all on the grid, so all the points marked are lattice points. The area of the shaded square is composed of the small inner square (which has whole-number sides) and the equivalent of two of the dotted rectangles (which also have whole-number sides). So, the shaded area is the sum of squares of whole numbers, which is also a whole number.